Indreprodukt

1o 9.16 16 17 23 24 30 F 3 Hanfs 21 22 Gfe } orig

Motivasjon:
- Generaliser vanlig skalarprodukt
- Tilpasset ∞-dim vektorrom / vektorer hvor noen koordinater er viktigere enn andre

Hva er ∞-dim vektorrom?

$\mathbb{R}^n \quad n \to \infty$

Eksempler:

Vektor koordinater $\mathbb{R}^3$ vektor $v_1, v_2, v_3 \quad v_i \geq 0$

Hvor like er to vektorer?

$$\bar{v} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\-1\end{pmatrix} \qquad \bar{w}_1 = \frac{1}{3}\begin{pmatrix}2\\2\\1\end{pmatrix} \qquad \bar{w}_2 = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

Hvilken er nærmest $\bar{v}$?

Hjelpemiddel: $\bar{v} \cdot \bar{w} = \|\bar{v}\| \cdot \|\bar{w}\| \cdot \cos\theta$ θ: vinkelen mellom

$$\bar{v} \cdot \bar{w}_1 = \frac{5}{3\sqrt{6}} \approx 0{,}68 \qquad \bar{v} \cdot \bar{w}_2 = \frac{1}{\sqrt{6}} \approx 0{,}41$$

$\bar{w}_1$ likner mest! (fordi hvis svaret er 1 er de helt like)

Vektor $(1, \frac{1}{2}, \frac{1}{6})$

$$\langle \bar{v}, \bar{w}_1 \rangle = \frac{1}{3\sqrt{6}}\left(1 \cdot 2 + \frac{1}{2} \cdot 2 + \frac{1}{6} \cdot 1 \cdot 1\right) = \frac{17}{6 \cdot 3\sqrt{6}} \approx 0{,}39$$

$$\langle \bar{v}, \bar{w}_2 \rangle = \frac{1}{\sqrt{6}}(1 \cdot 1 + 0) = \frac{1}{\sqrt{6}} \approx 0{,}41$$

Nå er $\bar{w}_2$ nærmest!

Vi kan til og med ha tyngdeskoblinger

To koordinater hver for seg: lav vekt
Se de i sammenheng: store utslag

Generell metode:

$$\langle \bar{v}, \bar{w} \rangle = \bar{v}^T Q \bar{w}$$

Q: 3×3-matrise

Ufravikelig krav: ~~$\bar{v}, \bar{w}$~~ $\langle \bar{v}, \bar{v} \rangle \geq 0$ $(\bar{v} \neq \bar{0} \Rightarrow \langle \bar{v}, \bar{v} \rangle > 0)$

Kalles: Q er positivt definit

Hvordan avgjøre om Q er positivt definit?

Indreprodukt

$\langle \bar{v}, \bar{w} \rangle = \bar{v}^T Q \bar{w}$ $\quad Q$ symmetrisk, positiv definit (semi+)

$\Updownarrow$

alle egenverdiene ≥ 0

Eksempel:

$$Q = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\chi_Q(\lambda) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16-12}}{2} = \begin{cases} 3 \\ 1 \end{cases}$$

$$\langle \bar{v}, \bar{w} \rangle = \bar{v}^T Q \bar{w} = 2v_1^2 - 2v_1v_2 + 2v_2^2$$

$$= v_1^2 + (v_1 - v_2)^2 + v_2^2 \geq 0$$

Annet eksempel på indreprodukt

A, B $n \times n$-matriser $\quad \langle A, B \rangle = \mathrm{tr}(A^T B)$

$$\left\langle \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right\rangle = \mathrm{tr}\left(\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right)$$

$$= \mathrm{tr} \begin{pmatrix} a_{11}b_{11} + a_{21}b_{21} & * \\ * & a_{12}b_{12} + a_{22}b_{22} \end{pmatrix}$$

$$= a_{11}b_{11} + a_{21}b_{21} + a_{12}b_{12} + a_{22}b_{22}$$

$$= (a_{11}\ a_{12}\ a_{21}\ a_{22}) \cdot (b_{11}\ b_{12}\ b_{21}\ b_{22})$$

Eksempel

$\bar{v}, \bar{w} \in \mathbb{R}^n$ $\quad \langle \bar{v}, \bar{w} \rangle = \bar{v}^T \begin{pmatrix} \frac{1}{n} & & 0 \\ & \frac{1}{n} & \\ 0 & & \frac{1}{n} \end{pmatrix} \bar{w}$

$$= \frac{1}{n} \bar{v}^T \bar{w}$$

$\bar{v} = \bar{w} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ gir $\langle \bar{v}, \bar{w} \rangle = \frac{1}{n} \cdot n = 1$ $\quad$ også når $n \to \infty$

Kan tenke på $\Delta x = \frac{1}{n}$, $\bar{v}(i) = v_i$

$$\Rightarrow \langle \bar{v}, \bar{w} \rangle = v(1)w(1) \cdot \Delta x + \cdots + v(n)w(n)\Delta x$$

$$\rightsquigarrow \quad \langle f, g \rangle = \lim_{\Delta x \to 0} \sum_{i=1}^{\frac{1}{\Delta x}} f(x_i) g(x_i) = \int_a^b f(x) g(x)\, dx$$

f, g
funksjoner $\mathbb{R} \to \mathbb{R}$

Indreprodukt:

- $\int_a^b \big(f_1(x) + f_2(x)\big) g(x)\, dx = \int_a^b f_1(x) g(x)\, dx + \int_a^b f_2(x) g(x)\, dx$
- $\int_a^b f(x) g(x)\, dx = \int_a^b g(x) f(x)\, dx$
- $\int_a^b f(x)^2\, dx \geq 0$ og $= 0$ hviss $f(x) = 0 \ \forall x$

Eksempel

$$\int_0^\pi \sin(x)\cos(x)\, dx = \left[\frac{1}{2}\sin^2 x\right]_0^\pi = 0$$

Deriverte av $\sin^2 x$: $2 \sin x \cos x$

$$\int_0^\pi \cos(x)\cos(x)\, dx = \int_0^\pi \frac{1+\cos 2x}{2}\, dx = \left[\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^\pi = \frac{\pi}{2}$$

$$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x)$$
$$= 2\cos^2 x - 1$$
$$\Rightarrow \cos^2 x = \frac{1+\cos 2x}{2}$$

$$\int_0^\pi \sin^2 x\, dx = \int_0^\pi 1 - \cos^2 x\, dx$$
$$= \pi - \int_0^\pi \cos^2 x\, dx$$
$$= \frac{\pi}{2}$$

som betyr at $\sqrt{\frac{2}{\pi}}\cos x$ har lengde 1 og står normalt på $\sqrt{\frac{2}{\pi}}\sin x$

ortonormal basis $\left\{\sqrt{\frac{2}{\pi}}\sin x, \sqrt{\frac{2}{\pi}}\cos x\right\}$

Eksempel

Polynomer av grad d er bestemt av verdien $d+1$ punkter

$p(x), q(x)$ polynomer av grad d

$$\langle p(x), q(x) \rangle = p(0)q(0) + p(1)q(1) + \dots + p(d)q(d)$$

Danner et indreprodukt

$P_1 = \{ a+bx \,,\ a,b \in \mathbb{R} \}$

Naturlig basis $\{1, x\}$ Men er den ortogonal/ortonormal?

$$\langle p(x), q(x) \rangle = p(0)q(0) + p(1)q(1)$$

$$\Rightarrow \quad \langle 1, 1 \rangle = 1\cdot 1 + 1\cdot 1 = 2 \Rightarrow \|1\| = \sqrt{2}$$

$$\langle x, x \rangle = 0\cdot 0 + 1\cdot 1 = 1 \Rightarrow \|x\| = 1$$

$$\langle 1, x \rangle = 1\cdot 0 + 1\cdot 1 = 1$$

Kan bruke Gram-Schmidt:

$$\bar{u}_1 = \frac{1}{\sqrt{2}}$$

$$\bar{u}_2 = x - \frac{\langle x, 1 \rangle}{\|1\|^2}\cdot 1 = x - \frac{1}{2}$$

$$\left\langle \frac{1}{\sqrt{2}}, x - \frac{1}{2} \right\rangle = \frac{1}{\sqrt{2}}\cdot\left(-\frac{1}{2}\right) + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = 0 \quad \underline{\text{ok}}$$

$$\|\bar{u}_2\|^2 = \langle \bar{u}_2, \bar{u}_2 \rangle = \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Basis ortonormal mhp $\langle , \rangle$:

$$\left\{ \frac{1}{\sqrt{2}}, \sqrt{2}\left(x - \frac{1}{2}\right) \right\}$$

Med et annet indreprodukt

$$\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)\,dx$$

$$\langle 1, 1 \rangle = \int_0^1 1\cdot 1\,dx = 1$$

$$\langle x, x \rangle = \int_0^1 x\cdot x\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$$

$$\langle 1, x \rangle = \int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$$

Gram-Schmidt

$$\bar{v}_1 = 1$$

$$\bar{v}_2 = x - \frac{\frac{1}{2}}{1}1 = x - \frac{1}{2}$$

$$\|\bar{v}_2\|^2 = \langle \bar{v}_2, \bar{v}_2 \rangle = \int_0^1 (x-\tfrac{1}{2})^2 dx$$

$$= \int_0^1 x^2 - x + \tfrac{1}{4}\, dx$$

$$= \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x\right]_0^1$$

$$= \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{4}{12} - \tfrac{6}{12} + \tfrac{3}{12} = \tfrac{1}{12}$$

$$\rightsquigarrow \quad \|\bar{v}_2\| = \sqrt{\tfrac{1}{12}} = \sqrt{\tfrac{1}{\sqrt{4\cdot 3}}} = \tfrac{1}{2\sqrt{3}}$$

$$\langle \bar{v}_1, \bar{v}_2 \rangle = \int_0^1 x - \tfrac{1}{2}\, dx = \left[\tfrac{1}{2}x^2 - \tfrac{1}{2}x\right]_0^1 = 0$$

$\{1,\ 2\sqrt{3}x - \sqrt{3}\}$ ortonormal basis

$$\int_0^1 (2\sqrt{3}x - \sqrt{3})^2 dx = \int_0^1 12x^2 - 12x + 3\, dx$$

$$= \left[\tfrac{12}{3}x^3 - \tfrac{12}{2}x^2 + 3x\right]_0^1$$

$$= \tfrac{12}{3} - \tfrac{12}{2} + 3 = 4 - 6 + 3 = \underline{\underline{1}}$$

Kan skrive dette indreprodukt som $\bar{v}^T Q \bar{w}$

$a + bx$ erstattes med $(a,b) \in \mathbb{R}^2$

$$\langle (a_1 + b_1 x), (a_2 + b_2 x) \rangle = a_1 a_2 \langle 1,1 \rangle + (a_1 b_2 + b_1 a_2) \langle 1, x \rangle + b_1 b_2 \langle x, x \rangle$$

$$= a_1 a_2 + (a_1 b_2 + b_1 a_2)\tfrac{1}{2} + b_1 b_2 \tfrac{1}{3}$$

$$= (a_1\ b_1) \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

svarer til $Q = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}$

$$\chi_Q(\lambda) = \begin{vmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3}-\lambda \end{vmatrix} = \lambda^2 - \tfrac{4}{3}\lambda + (\tfrac{1}{3} - \tfrac{1}{4}) = \lambda^2 - \tfrac{4}{3}\lambda + \tfrac{1}{12}$$

abc: $\lambda = \dfrac{\frac{4}{3} \pm \sqrt{\frac{16}{9} - \frac{4}{12}}}{2}$

$$= \tfrac{2}{3} \pm \tfrac{1}{2}\sqrt{\tfrac{12}{9}} = \tfrac{2}{3} \pm \tfrac{\sqrt{3}}{6}$$

$$= \tfrac{4 \pm \sqrt{3}}{6} > 0.$$

Husk: Hvorfor er vi interessert i ortonormal basis?

$$\mathcal{B} = \{\bar{u}_1, \bar{u}_2\}$$

ortonormal

$$\langle \bar{u}_1, \bar{u}_1 \rangle = \langle \bar{u}_2, \bar{u}_2 \rangle = 1$$

$$\langle \bar{u}_1, \bar{u}_2 \rangle = 0$$

$\bar{v} \in \mathbb{R}^2$ vilkårlig $\Rightarrow$ $\bar{v} = \langle \bar{v}, \bar{u}_1 \rangle \cdot \bar{u}_1 + \langle \bar{v}, \bar{u}_2 \rangle \bar{u}_2$