Indreprodukt

Repetisjon:

Eksempel $V = \text{Span}(e^x, e^{-x})$

Indreprodukt $\langle f, g \rangle = \int_0^1 f \cdot g \, dx$

$$\langle e^x, e^x \rangle = \int_0^1 e^{2x} dx = \left[\tfrac{1}{2}e^{2x}\right]_0^1 = \tfrac{1}{2}(e^2 - e^0) = \tfrac{1}{2}(e^2 - 1)$$

$$\Rightarrow \|e^x\| = \sqrt{\tfrac{e^2-1}{2}}$$

$$\langle e^{-x}, e^{-x} \rangle = \int_0^1 e^{-2x} dx = \left[-\tfrac{1}{2}e^{-2x}\right]_0^1 = -\tfrac{1}{2}(e^{-2} - e^0) = \tfrac{1-e^{-2}}{2}$$

$$\|e^{-x}\| = \sqrt{\tfrac{1-e^{-2}}{2}}$$

$$\langle e^x, e^{-x} \rangle = \int_0^1 e^x \cdot e^{-x} dx = \int_0^1 1 \, dx = 1$$

$$\langle ae^x + be^{-x}, ce^x + de^{-x} \rangle = ac\,\tfrac{1}{2}(e^2 - 1) + (ad + bc) + bd\,\tfrac{1}{2}(1 - e^{-2}) = 0$$

Kan for eksempel sette $a = c = \sqrt{\tfrac{1-e^{-2}}{2}} = \|e^{-x}\|$

$-b = d = \sqrt{\tfrac{e^2-1}{2}} = \|e^x\|$

$$= \tfrac{1-e^{-2}}{2}\tfrac{e^2-1}{2} + 0 - \tfrac{e^2-1}{2}\tfrac{1-e^{-2}}{2} = 0.$$

$$\{\|e^{-x}\| e^x - \|e^x\| e^{-x},\ \|e^{-x}\| e^x + \|e^x\| e^{-x}\}$$

ortogonal basis

$$\| \|e^{-x}\| e^x - \|e^x\| e^{-x} \|^2 = \|e^{-x}\|^2 \|e^x\|^2 - 2\|e^{-x}\|\|e^x\| + \|e^x\|^2 \|e^{-x}\|^2$$

$$= 2\|e^{-x}\|\|e^x\| (\|e^{-x}\|^2 \|e^x\| - 1)$$

$$\tfrac{e^2-1}{2}\,\tfrac{1-e^{-2}}{2} = \tfrac{1}{4}(e^2 - 1 + 1 - e^{-2}) = \tfrac{1}{4}(e^2 - e^{-2})$$

Eksempel Hvis vi har komplekse koordinater må vi modifisere indreproduktet: $V = \mathbb{C}$

$$\langle z_1, z_2 \rangle = z_1 \cdot \overline{z_2}$$

fordi $(a+ib)(a+ib) = a^2 + 2iab - b^2$ Ikke noen lengde!

$$(a+ib)\cdot(a-ib) = a^2 + b^2 \geq 0$$

og $= 0$ dersom $a = b = 0$

$V = \text{Span}(e^{ix}, e^{-ix})$ $\quad \langle f, g\rangle = \int_0^{\pi} f \cdot \overline{g}\, dx$

gir $\|e^{ix}\|^2 = \int_0^{\pi} e^{ix} \cdot e^{-ix}\, dx = \pi$.

$$\langle e^{ix}, e^{-ix}\rangle = \int_0^{\pi} e^{ix} e^{ix}\, dx = \int_0^{\pi} e^{2ix}\, dx$$

$$= \left[\frac{1}{2i} e^{2ix}\right]_0^{\pi} = \frac{1}{2i}\left(e^{2\pi i} - e^{0}\right) = 0.$$

Merk at i dette tilfellet faller vi

$$\langle f, g\rangle = \overline{\langle g, f\rangle}$$ Nyttig, men ikke pensum

Veldig viktig med ortogonal basis.

$\mathcal{B} = \{\bar{v}_1, \dots, \bar{v}_n\}$ ortogonal basis for V, $\bar{w} \in V$

$$\Rightarrow \bar{w} = \frac{\langle \bar{w}, \bar{v}_1\rangle}{\|\bar{v}_1\|^2}\bar{v}_1 + \dots + \frac{\langle \bar{w}, \bar{v}_n\rangle}{\|\bar{v}_n\|^2}\bar{v}_n.$$

forde

$$\left\langle \bar{w} - \frac{\langle \bar{w}, \bar{v}_1\rangle}{\|\bar{v}_1\|^2}\bar{v}_1 - \dots - \frac{\langle \bar{w}, \bar{v}_n\rangle}{\|\bar{v}_n\|^2}\bar{v}_n, \bar{v}_i \right\rangle$$

$$= \langle \bar{w}, \bar{v}_i\rangle - \left\langle \frac{\langle \bar{w}, \bar{v}_i\rangle}{\|\bar{v}_i\|^2}\bar{v}_i \right\rangle$$

$$= \langle \bar{w}, \bar{v}_i\rangle - \langle \bar{w}, \bar{v}_i\rangle \frac{\|\bar{v}_i\|^2}{\|\bar{v}_i\|^2} = 0.$$

Hvis $\mathcal{B}$ er ortonormal; $\|\bar{v}_i\| = 1$

$$\bar{w} = \langle \bar{w}, \bar{v}_1\rangle \bar{v}_1 + \dots + \langle \bar{w}, \bar{v}_n\rangle \bar{v}_n$$

(Hvis ikke ortogonal: $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$, men $\|\bar{v}_i\| = 1$.

1. Gram-Schmidt $\quad \bar{u}_1 = \bar{v}_1$

$$\bar{u}_2 = v_2 - \langle \bar{v}_2, \bar{u}_1\rangle \bar{v}_1 / \| \|$$

$$\bar{u}_3 = v_3 - \langle \bar{v}_3, \bar{v}_1\rangle \bar{u}_1 - \langle \bar{v}_3, \bar{v}_2\rangle \bar{u}_2 / \| \|$$

2. Dekomponer som over

Antall indreprodukt utregninger. 1. $1+2+3+\dots+n = \frac{n(n+1)}{2}$

2. $2n$ / $2n$

	1	2	3	4	5	6	7	8	9
ortogonal	2	4	6	8	10	12	14	16	18
ikke ortogonal	2	5	9	14	20	27	38	44	54
	lik				dobbel				tippel

$$n + \frac{1}{2}n^2 + \frac{n}{2} = \frac{1}{2}(n^2+3n)$$

V.S

$2n$

$n = 1000$: $\frac{1}{2}(1000^2 + 3\cdot 1000) \approx 500\,000$

$2n = 2000$

Eksempel: Frobenius produktet

$A, B \in M_n \qquad \langle A, B\rangle = tr(A^T B)$

$n=2$ $\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\-1&0\end{pmatrix}\right\}$

ortogonal basis
(hver har lengde $\sqrt{2}$)

$$\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix} = \frac{a_{11}+a_{22}}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} + \frac{a_{11}-a_{22}}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$
$$+ \frac{a_{12}+a_{21}}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} + \frac{a_{12}-a_{21}}{2}\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

3 P_2 : polynomes au grad ≤ 2

$$\langle f(x), g(x) \rangle = f(-1)g(-1) + f(0)g(0) + f(1)g(1)$$

a) $\left\langle \frac{1}{\sqrt{2}}x, \frac{1}{\sqrt{2}}x \right\rangle = -\frac{1}{\sqrt{2}} \cdot -\frac{1}{\sqrt{2}} + 0 \cdot 0 + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2} + \frac{1}{2} = 1.$

$\langle x^2-1, x^2-1 \rangle = 0 \cdot 0 + (-1)(-1) + 0 \cdot 0 = 1.$

$\left\langle \frac{1}{\sqrt{2}}x, x^2-1 \right\rangle = -\frac{1}{\sqrt{2}} \cdot 0 + 0(-1) + \frac{1}{\sqrt{2}} 0 = 0$

4. $D_6 = \langle S, R \mid S^2 = R^6 = id,\ SR^5 = RS \rangle \longrightarrow GL(\mathbb{R}^2)$

$$\rho = \varphi(R) = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

$$\mu = \varphi(S) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

a) $\rho^2 = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$ $\rho^3 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\Rightarrow$ ρ <u>har orden 6.</u>

$\mu^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

b) $\mu\rho^5 = \rho\mu \Leftrightarrow \mu = \rho\mu\rho$

$$\rho\mu\rho = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \mu$$

Alternativ: $\rho = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

forda $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\Rightarrow \rho^5 = \rho^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ $\mu\rho^5 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & +\frac{1}{2} \end{pmatrix}$

$$\rho\mu = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Oppgavesett 1.

1. $M = \begin{pmatrix} 0,6 & 0,2 & 0,2 \\ 0,1 & 0,5 & 0,4 \\ 0,3 & 0,3 & 0,4 \end{pmatrix} \quad \bar{x}_0 = \begin{pmatrix} 300 \\ 0 \\ 0 \end{pmatrix}$

a) $\bar{x}_1 = M \cdot \bar{x}_0 = \begin{pmatrix} 180 \\ 30 \\ 90 \end{pmatrix}$

b) $\chi_M(\lambda) = \begin{vmatrix} 0,6-\lambda & 0,2 & 0,2 \\ 0,1 & 0,5-\lambda & 0,4 \\ 0,3 & 0,3 & 0,4-\lambda \end{vmatrix} = -\lambda^3 + 1,5\lambda^2 - 0,54\lambda + 0,04$

$= -(\lambda-1)(\lambda^2 - 0,5\lambda + 0,04)$

$= -(\lambda-1)(\lambda-0,4)(\lambda-0,1) = 0$

Egenverdia $\lambda_1 = 1 \quad \lambda_2 = 0,4 \quad \lambda_3 = 0,1$

c) Egenvektor $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

d) $\bar{u} = \begin{pmatrix} 100 \\ 100 \\ 100 \end{pmatrix}$

e) $\begin{pmatrix} 300 \\ 0 \\ 0 \end{pmatrix} = 100 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 200 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 100 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

f) $\lim_{n\to\infty} \bar{x}_n = \begin{pmatrix} 100 \\ 100 \\ 100 \end{pmatrix} + \lim_{n\to\infty} 0,4^n \cdot 200 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lim_{n\to\infty} (0,1)^n \cdot 100 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 100 \\ 100 \\ 100 \end{pmatrix}$

2. $\bar{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \mapsto -\bar{v} \qquad \bar{w} \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto -\bar{w}$

a) $\left.\begin{matrix} x+y = 0 \\ z = 0 \end{matrix}\right\} \quad x = -y, z = 0 \quad (\text{langs } \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix})$

b) $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\det A = 1$ Rotasjon om z-aksen

c) $A^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A^4 = \text{id}$ orden 4

Sett 2

oppgave 1

$$T(\bar{x}) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\bar{x} + \begin{pmatrix}0\\ 1\end{pmatrix}$$

$\det\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\right) = 1$ Rotasjon

$$T(\bar{x}) = \bar{x} \Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\bar{x} + \begin{pmatrix}0\\ 1\end{pmatrix} = \bar{x}$$

$$\bar{x} = \begin{pmatrix}x\\ y\end{pmatrix} \Rightarrow \begin{aligned}\tfrac{1}{\sqrt{2}}x + \tfrac{1}{\sqrt{2}}y &= x\\ -\tfrac{1}{\sqrt{2}}x + \tfrac{1}{\sqrt{2}}y + 1 &= y\end{aligned}$$

$$\Rightarrow \left(\tfrac{1}{\sqrt{2}}-1\right)x + \tfrac{1}{\sqrt{2}}y = 0 \quad \left\} \; (1-\sqrt{2})x + y = 0\right.$$
$$-\tfrac{1}{\sqrt{2}}x + \left(\tfrac{1}{\sqrt{2}}-1\right)y = -1 \quad \left\} \; -x + (1-\sqrt{2})y = -\sqrt{2}\right.$$

$1 + (1-\sqrt{2})(\sqrt{2}-1)$ Fortsatt bra tid stede

2. a)

$$\begin{pmatrix}0{,}5 & 0{,}1\\ 0{,}5 & 0{,}9\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x\\ y\end{pmatrix} \qquad \begin{aligned}-0{,}5x + 0{,}1y &= 0\\ 0{,}5x - 0{,}1y &= 0\end{aligned}$$

$$\chi_M(\lambda) = \begin{vmatrix}0{,}5-\lambda & 0{,}1\\ 0{,}5 & 0{,}9-\lambda\end{vmatrix} = \begin{vmatrix}0{,}5 & 0{,}9\\ 0{,}5 & 0{,}1\end{vmatrix} =$$

$$= \lambda^2 - 1{,}4\lambda + 0{,}4 = 0 \Rightarrow \lambda = \frac{1{,}4 \pm \sqrt{1{,}96 - 1{,}6}}{2}$$

$$= \frac{1{,}4 \pm 0{,}6}{2} = \begin{cases}1\\ +0{,}4\end{cases}$$

$\lambda = 1$ $\begin{pmatrix}x\\ y\end{pmatrix} = \text{Span}\left\{\begin{pmatrix}1\\ 5\end{pmatrix}\right\}$

$\lambda = -0{,}4$ $\begin{pmatrix}x\\ y\end{pmatrix} = \text{Span}\left\{\begin{pmatrix}1\\ -9\end{pmatrix}\right\}$ $\begin{pmatrix}0{,}5 & 0{,}1\\ 0{,}5 & 0{,}9\end{pmatrix}\begin{pmatrix}1\\ -9\end{pmatrix} = \begin{pmatrix}+0{,}4\\ -0{,}4\end{pmatrix}$

b) $\begin{pmatrix}150\\ 150\end{pmatrix} = x\begin{pmatrix}1\\ 5\end{pmatrix} + y\begin{pmatrix}1\\ -1\end{pmatrix} = \begin{pmatrix}x+y\\ 5x-y\end{pmatrix}$ $\begin{aligned}x+y &= 150\\ 5x-y &= 150\end{aligned}$

$\Rightarrow 6x = 300 \quad x = 50$

$\Rightarrow y = 100.$

$\begin{pmatrix}150\\150\end{pmatrix} = \begin{pmatrix}50\\250\end{pmatrix} + \begin{pmatrix}100\\-100\end{pmatrix}$ $\quad M^n\begin{pmatrix}150\\150\end{pmatrix} = \begin{pmatrix}50\\250\end{pmatrix} + 0{,}4^n\begin{pmatrix}100\\-100\end{pmatrix} \to \begin{pmatrix}50\\250\end{pmatrix}$

3. a) $A = \begin{pmatrix}1&-2\\1&0\end{pmatrix}$ $B = \begin{pmatrix}1&1\\1&0\end{pmatrix}$

$\|A\|^2 + \|B\|^2 = \|A-B\|^2$

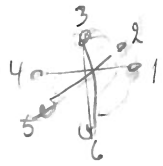
$tr\left(\begin{pmatrix}1&1\\-2&0\end{pmatrix}\begin{pmatrix}1&-2\\1&0\end{pmatrix}\right) + tr\left(\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\1&0\end{pmatrix}\right) = tr\left(\begin{pmatrix}0&0\\-3&0\end{pmatrix}\begin{pmatrix}0&-3\\0&0\end{pmatrix}\right)$

$2 + 4 + 2 + 1 = 9$

b) $tr(M^T) = tr(M) \quad tr\,M.$

c) $tr(M^TN) = tr((M^TN)^T) = tr(N^TM)$

$\langle M, N\rangle = \langle N, M\rangle$

4. 

Rotasjon $\frac{\pi}{2}$ om hver akse.

Refleksjon om koordinat planen

— " — om $x=y$, $x=z$ og $y=z$.

a) $S_x = \begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}$ $S_y = \begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix}$ symmetri

b) $S_z = \begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}$ $S_x: \begin{matrix}x\mapsto x\\ y\mapsto z\\ z\mapsto -y\end{matrix}$ $S_y \begin{matrix}x\mapsto -z\\ y\mapsto y\\ z\mapsto x\end{matrix}$

$S_x^{-1}S_y^{-1}S_x = \begin{matrix}x\mapsto x\mapsto z\mapsto y\\ y\mapsto z\mapsto -x\mapsto -x\\ z\mapsto -y\mapsto -y\mapsto +z\end{matrix}$ $\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}$

$x\to x\mapsto z\mapsto y$
$y\to +z\mapsto x\mapsto -x$
$z\to -y\mapsto -y\mapsto +z$

c) $S_z = S_x^{-1}S_y^{-1}S_x = \begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}\begin{pmatrix}0&0&-1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix} = \begin{pmatrix}0&0&-1\\1&0&0\\0&-1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix} = \begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}$

$x\mapsto -z\mapsto y\mapsto y$
$y\mapsto y\mapsto z\mapsto -x$
$z\mapsto x\mapsto -x\mapsto z$

$S_z = S_y^{-1}S_xS_y = \begin{pmatrix}0&1&0&-1\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&0&-1\\0&0&1&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&1&0&0\\-1&0&0&0\\0&0&0&1\end{pmatrix} = \begin{pmatrix}0&-1&0&0\\0&0&0&-1\\1&0&0&0\\0&0&1&0\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&1&0&0\\-1&0&0&0\\0&0&0&1\end{pmatrix} = \begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$

$\Rightarrow S_x^{-1}S_y^{-1}S_x = S_y^{-1}S_xS_y$

$S_x = (2356)$
$S_y = (1643)$
$S_z = (1245)$

$(2653)(1346)(2356) = (1245)$
$(1346)(2356)(1643) = (1245)$

Fourier-rekker

Studere bølgefunksjonen $\sin(n\omega t)$ $\cos(n\omega t)$

For enhetens skyld settes $\omega = 1$.



Illustrasjon: svingende streng med overtoner

Lyder er satt sammen av overtoner med ulik amplitude

Eks $\sin(t) + 0{,}1\sin(2t) + 0{,}3\sin(3t) + \ldots$ etc.

$\sin(t)$: grunntone

overtoner: amplituden avtar med økende n.

Vårt univers er vektorrommet (B)

$$V = \text{Span}\{1, \sin t, \cos t, \sin 2t, \cos 2t, \sin 3t, \cos 3t, \ldots\}$$

∞-dimensjonalt

Indreprodukt $\langle f(t), g(t)\rangle = \int_0^{2\pi} f(t)\,g(t)\,dt$

Dette kan vi regne ut:

$$\int_0^{2\pi} \cos(nt)\,dt = \begin{cases} 0 & n \neq 0 \\ 2\pi & n = 0 \end{cases}$$

$$\int_0^{2\pi} \sin(nt)\,dt = 0 \quad \forall\, n \geq 0$$

$$\int_0^{2\pi} \cos(mt)\cos(nt)\,dt = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$

$$\int_0^{2\pi} \sin(mt)\sin(nt)\,dt = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$

$$\int_0^{2\pi} \sin(mt)\cos(nt)\,dt = 0 \quad \forall\, n \geq 0$$

Observasjon: $\mathcal{B}$ er en ortogonal basis for V.

Bølgefunksjon

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt)\right)$$

Bruker ortogonaliteten og finner ved å gå på

$$\langle f(t), \cos(nt)\rangle \text{ og } \langle f(t), \sin(nt)\rangle$$

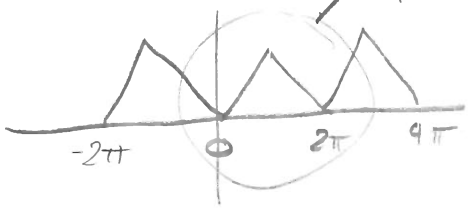
$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t)\cos(nt)\,dt$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t)\sin(nt)\,dt$$

$$\frac{a_0}{2} = \int_0^{2\pi} f(t)\,dt.$$

Eks

Sagtann-kurven



beskrevet ved

$$f(t) = \begin{cases} t & 0 \le t \le \pi \\ 2\pi - t & \pi \le t \le 2\pi \end{cases}$$

Utregning

$$b_n = 0 \quad n \ge 1$$

$$a_n = \begin{cases} \pi & n = 0 \\ 0 & n \neq 0 \text{ og partall} \\ -\frac{4}{\pi n^2} & n \text{ oddetall} \end{cases}$$

$$f(t) = \frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{4}{\pi (2k-1)^2} \cos((2k-1)t)$$

Eksempel



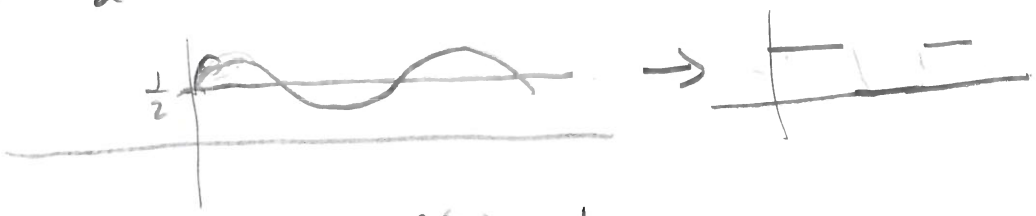
$$f(t) = \begin{cases} 0 & (2k-1)\pi \leq t \leq 2k\pi \\ 1 & 2k\pi < t \leq (2k+1)\pi \end{cases}$$

Utregning gir $a_0 = 1$

$a_n = 1.$

$$b_n = -\frac{(-1)^n}{n\pi} + \frac{1}{n\pi}$$

$$\Rightarrow f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \dots\right)$$



sette nn $t = \frac{\pi}{2}$ Vet at $f\left(\frac{\pi}{2}\right) = 1.$

$$\Rightarrow 1 = \frac{1}{2} + \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots\right)$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$$

oppdaget av James Gregory i 1668.

$\rightsquigarrow$ Kan beregne π!!!