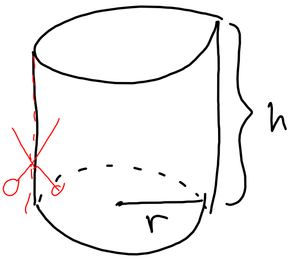
OBLIG 1 Lagt ut:

Eksempel:



Materiale: $100\,cm^2$

Ønsker størst mulig volum.

$V = g\,h = \pi r^2 h$

Arealbegrensning:

$$100 = A = \underbrace{2\pi r^2}_{\text{Lunnatopp}} + \underbrace{2\pi r h}_{\text{sideflaten}}$$



Løser for h: $2\pi r h = 100 - 2\pi r^2$

$$h = \frac{100 - 2\pi r^2}{2\pi r}$$

$$V = \pi r^2 h = \cancel{\pi r^2}\,\frac{100 - 2\pi r^2}{\cancel{2\pi r}} = \frac{1}{2}\left(100r - 2\pi r^3\right)$$

Deriverer:

$$V' = \frac{1}{2}\left(100 - 6\pi r^2\right)$$

Finner nå når $V'(r) = 0$:

$$6\pi r^2 = 100$$

$$r^2 = \frac{100}{6\pi} \Rightarrow r = \frac{\sqrt{100}}{\sqrt{6\pi}} = \frac{10\cdot\sqrt{6}}{\sqrt{6\pi}\cdot\sqrt{6}} = \frac{10\sqrt{6}}{6\pi}$$

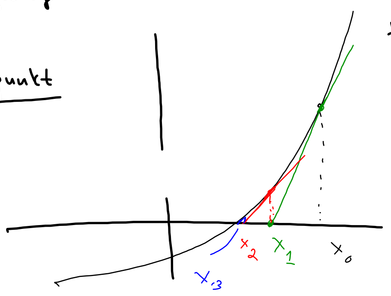
$$= \frac{5}{3}\frac{\sqrt{6}}{\pi}$$ Maksimal.

Newtons metod

Finna nollpunkt: $f(x) = 0$

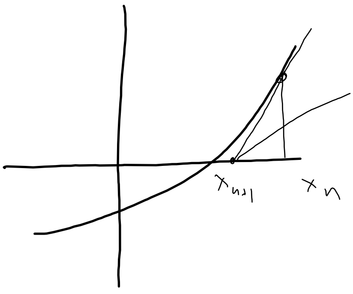
Hur ligningen ikke kan løses eksakt, så trenger vi tilnærmede løsninger.

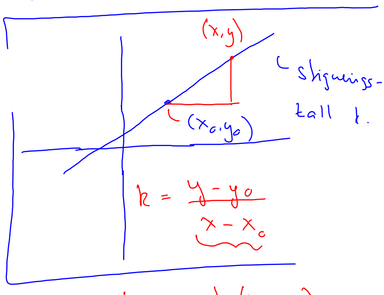
Utgangspunkt



Tangenten til $y = f(x)$ i punktet x_0:

$$y - f(x_0) = f'(x_c)(x - x_c)$$





$y - y_0 = k(x - x_0)$

Tangentligning: $y - f(x_n) = f'(x_n)(x - x_n)$

Setter inn $(x_{n+1}, 0)$ for (x, y):

$$0 - f(x_n) = f'(x_n)(x_{n+1} - x_n)$$

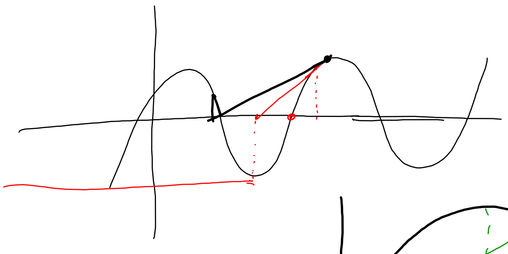
Løser for x_{n+1}:

$$-\frac{f(x_n)}{f'(x_n)} = x_{n+1} - x_n$$

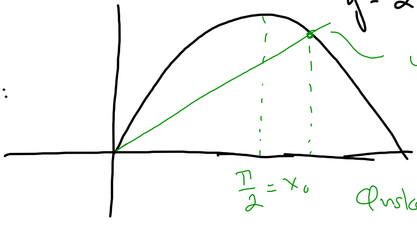
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Newtons metode: Velg x_0, deretter regn ut

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



Eksempel:



$2\sin x = x$

Ønsker tilnærmet løsning.

$f(x) = 2\sin x - x$ ← vil finne nullpunkt

Newtons metode: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2\sin x_n - x_n}{2\cos x_n - 1}$

Velger $x_0 = \frac{\pi}{2} \approx 1.5708$

$f'(x) = 2\cos x - 1$

$$x_1 = x_0 - \frac{2\sin x_0 - x_0}{2\cos x_0 - 1} = 1.9010$$

$$x_2 = x_1 - \frac{2\sin x_1 - x_1}{2\cos x_1 - 1} = 1.8955\ldots$$

$$x_3 = x_2 - \frac{2\sin x_2 - x_2}{2\cos x_2 - 1} = 1.8955\ldots$$

Tilnærmet løsning

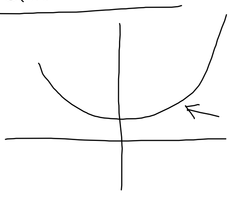
1.8955

$f(1.8955) = 0\ldots$

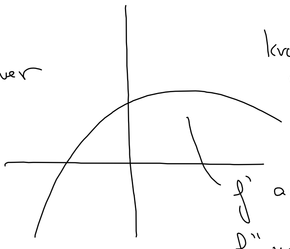
KRUMNING

Annenderivata: $(f')' = f''$

Geometrisk:

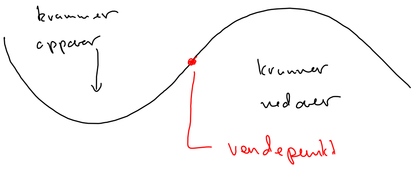


krummer oppover
konveks
vokser
f'' er positiv



krummer nedover
konkav
f' avta
f'' negativ

Krumningen kan skifte:



krummer oppover
krummer nedover
vendepunkt

Oppsummering: Hvis $f''(x) \geq 0$ på et intervall I, så krummer funksjonen oppover på I.

Hvis $f''(x) \leq 0$ på et intervall I, så krummer funksjonen nedover på I.

Et punkt hvor funksjonen skifter krumningsvei, kalles et vendepunkt

Eksempel: $f(x) = e^{-x^2}$ Hvordan krummer denne?

Deriver: $f'(x) = e^{-x^2} \cdot (-2x) = -2x e^{-x^2}$

Deriver en gang til:

$$f''(x) = -2e^{-x^2} + (-2x)e^{-x^2}(-2x)$$

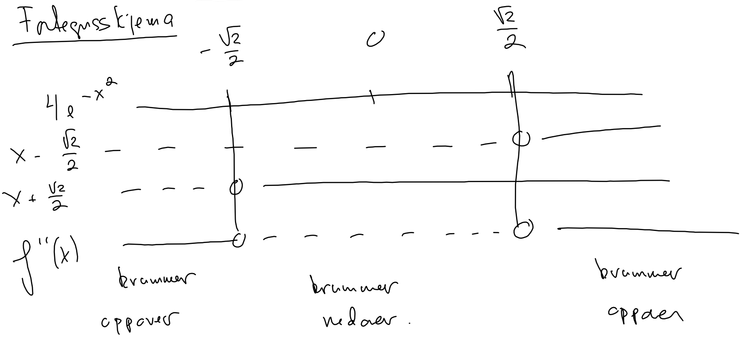
$$= e^{-x^2}(-2 + 4x^2) = 4e^{-x^2}(x^2 - \tfrac{1}{2})$$

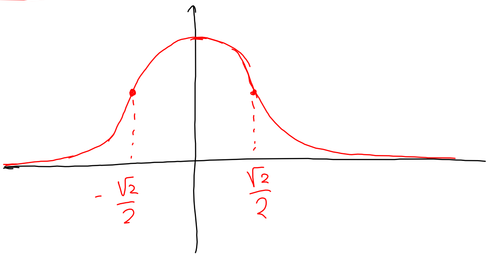
$$= 4e^{-x^2}\left(x - \frac{\sqrt{2}}{2}\right)\left(x + \frac{\sqrt{2}}{2}\right)$$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{1}}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$

Fortegnsskjema





L'Hôpitals regel

Hvordan regner vi ut $\lim_{x \to a} \frac{f(x)}{g(x)}$ når $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$

eller $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty$

Eksempel: $\lim_{x \to 0} \frac{\sin x}{x} = 1$

L'Hôpitals regel: Dersom

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$$

eller

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty, \text{ så er}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

forutsatt at den siste grenseverdien eksisterer (i godtar at den er $\pm \infty$)

Eksempel: $\lim_{x \to 0} \frac{1 - e^x}{x} \overset{L'H}{=} \lim_{x \to 0} \frac{0 - e^x}{1} = \frac{-1}{1} = -1$

Eksempel: $\lim_{x \to 0} \frac{\tan x}{x} \overset{L'H}{=} \lim_{x \to 0} \frac{\frac{1}{\cos^2 x} \to 1}{1} = 1$

Eksempel: $\lim_{x \to 0} \frac{1 - \cos x}{x^2} \overset{L'H}{=} \lim_{x \to 0} \frac{\sin x}{2x} \overset{L'H}{=} \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$

Eksempel: $\lim_{x \to 0^+} x \ln x$ ($x \to 0$, $\ln x \to -\infty$)

$$= \lim_{x \to 0^+} \frac{x \ln x}{1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \quad (\ln x \to -\infty, \ \frac{1}{x} \to \infty)$$

$$\overset{L'H}{=} \lim_{x \to 0^+} \frac{\left(\frac{1}{x}\right) x^2}{\left(-\frac{1}{x^2}\right) x^2} = \lim_{x \to 0^+} \frac{x}{-1} = 0$$

Eksempel: $\lim_{x \to \infty} \left(\sqrt{x^2 + x} - x \right)$ ($\sqrt{x^2+x} \to \infty$, $-x \to -\infty$)

$$= \lim_{x \to -\infty} \left(x \sqrt{1 + \frac{1}{x}} - x \right)$$

$$= \lim_{x \to -\infty} x \left(\sqrt{1 + \frac{1}{x}} - 1 \right) \quad (x \to -\infty, \ \sqrt{1+\frac{1}{x}} - 1 \to 0)$$

$$= \lim_{x \to -\infty} \frac{\sqrt{1 + \frac{1}{x}} - 1}{\frac{1}{x}}$$

$$\overset{L'H}{=} \lim_{x \to -\infty} \frac{\frac{1}{2\sqrt{1+\frac{1}{x}}} \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \frac{1}{2}$$

$a\sqrt{b} = \sqrt{a^2 b}$

$\sqrt{a^2}\sqrt{b} = \sqrt{a^2 b}$

Hvarför en L'Hôpital viktig?

Beviset i tillfället hvar $\lim_{x\to a} f(x) = f'(a)$ och $\lim_{x\to a} g'(x) = g'(a)$ är konstgjort för 0.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

($f(a) = 0$, $g(a) = 0$; $\frac{f(x)-f(a)}{x-a} \to f'(a)$, $\frac{g(x)-g(a)}{x-a} \to g'(a)$)

$$= \frac{f'(a)}{g'(a)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$