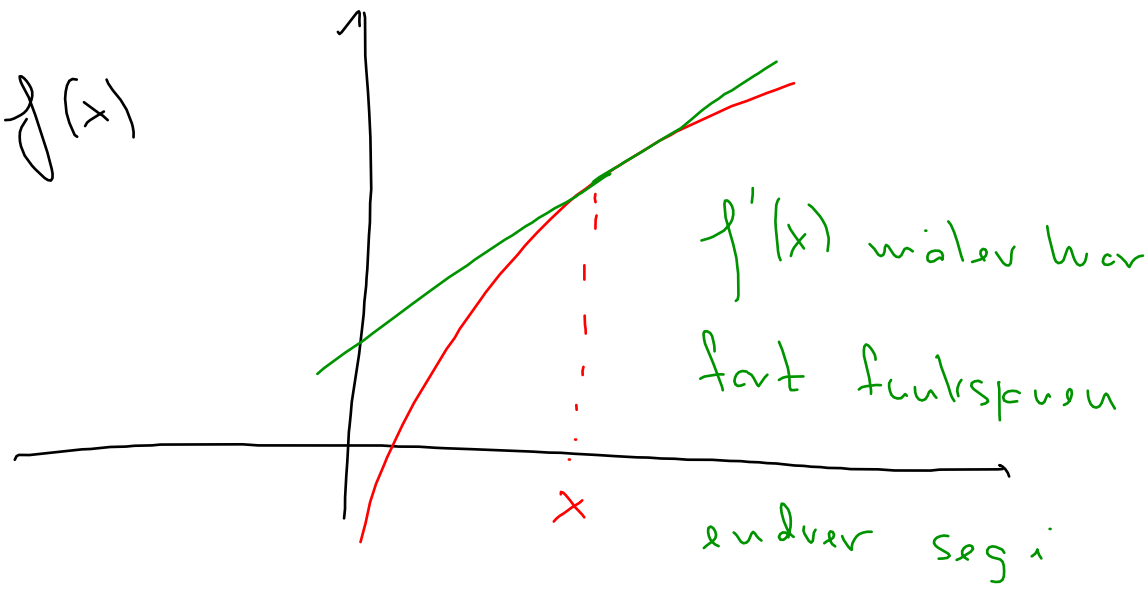
Partiell derivation

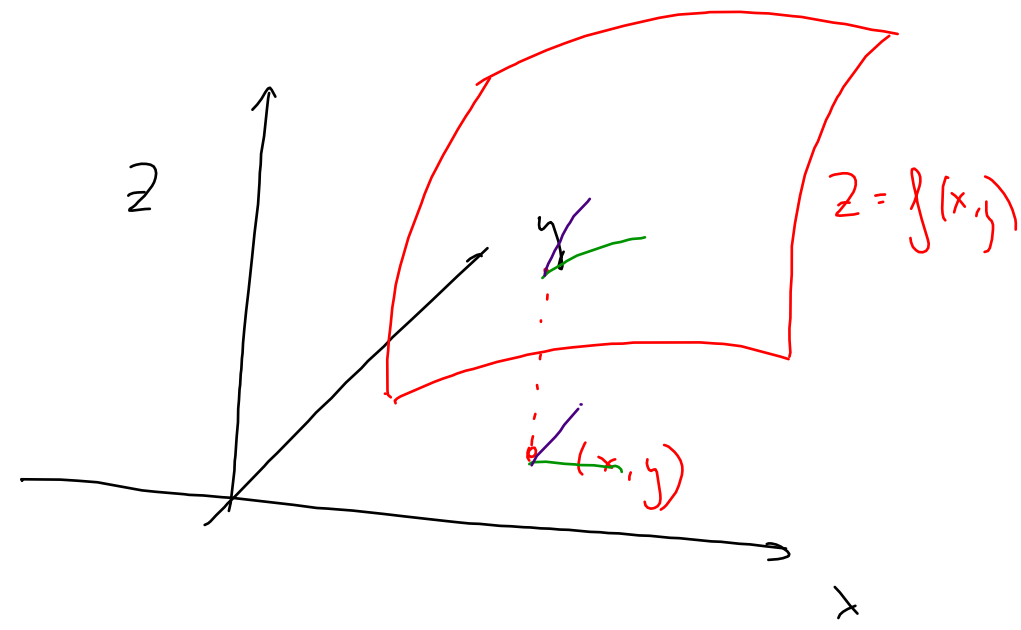
Vanlig derivation: $y = f(x)$



$$f(x+\Delta x) - f(x) \approx f'(x)\,\Delta x$$

Två variabler

$$z = f(x,y)$$



Ändringskvoten avhänger av riktningen.

Ändring i x-riktning

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \frac{\partial f}{\partial x}(x,y)$$

partialderivata i x-riktning

Ändring i y-riktning

$$\lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = \frac{\partial f}{\partial y}(x,y)$$

partialderivata i y-riktning

Observation: När vi deriverar mhp en variabel, fungerar den andra variabeln som en konstant.

$$f(x,y) = x^2 y^3$$

$\frac{\partial f}{\partial x}$ = "derivera mhp x som om y var en konstant"

$= 2xy^3$

$\frac{\partial f}{\partial y}$ = "derivera mhp y som om x var en konstant"

$= x^2 3y^2 = 3x^2 y^2$

Exempel: $f(x,y) = x \sin(\underbrace{xy^2}_{\text{hjälp}})$

$$\frac{\partial f}{\partial x} = 1 \cdot \sin(xy^2) + x \cos(xy^2) \cdot y^2 = \sin(xy^2) + xy^2 \cos(xy^2)$$

$$\frac{\partial f}{\partial y} = x \cos(xy^2)(x\,2y) = 2x^2 y \cos(xy^2)$$

Funksjoner av n variable

$$y = f(x_1, x_2, \ldots, x_n)$$

x_i

$$\frac{\partial f}{\partial x_i} = \lim_{\Delta x \to 0} \frac{f(x_1, \ldots, x_i + \Delta x, \ldots, x_n) - f(x_1, x_2, \ldots x_i, \ldots, x_n)}{\Delta x}$$

partiell derivert i x_i - retning.

Eksempel: $f(x,y,z) = x^2 z + \underbrace{xyz}_{\text{hjerne}}$

$xz + y$

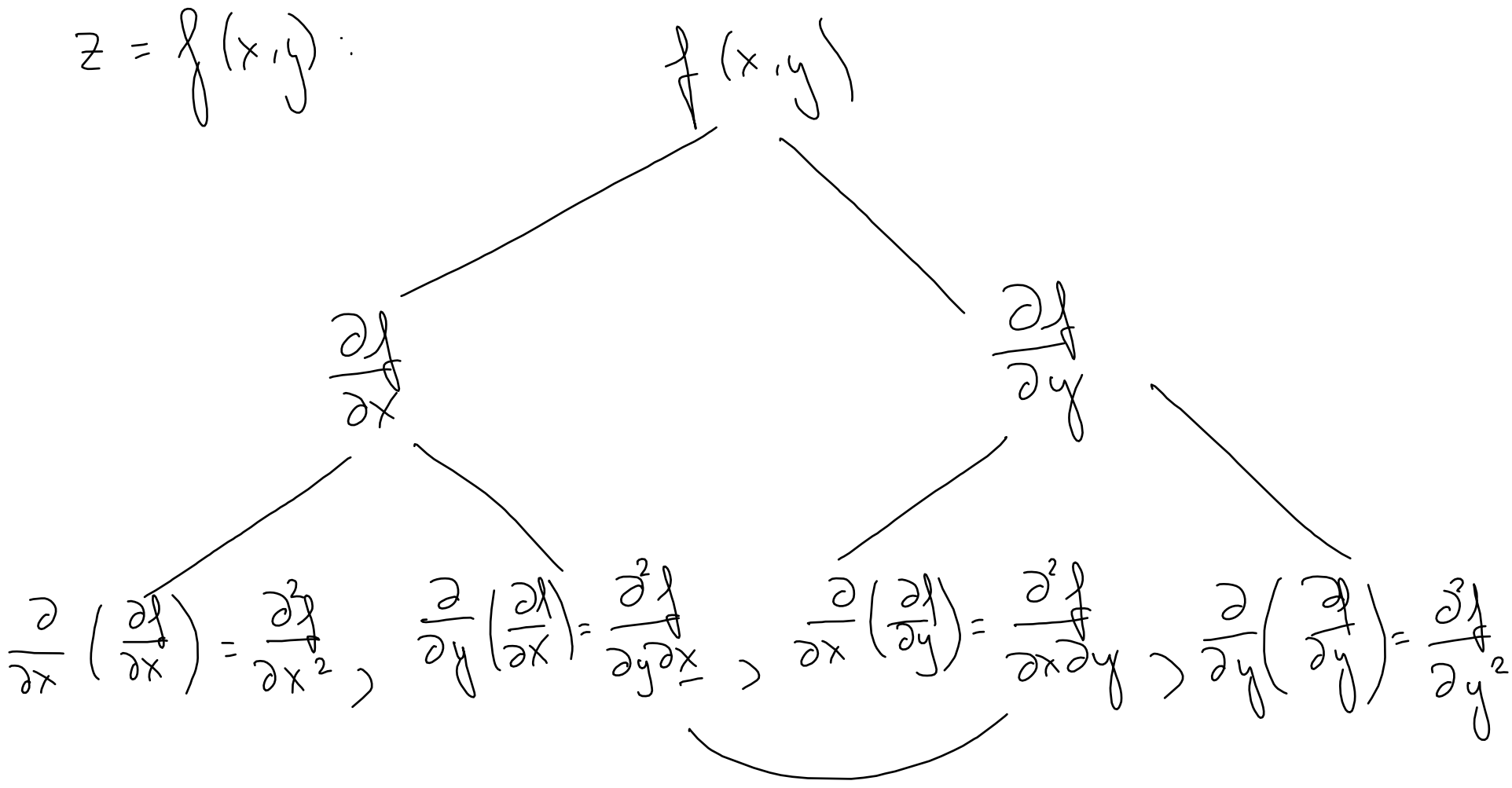
$$\frac{\partial f}{\partial x} = 2xz + 1 \cdot y e^{xz+y} + xy e^{xz+y} \cdot z = 2xz + e^{xz+y}(y + xyz)$$

$$\frac{\partial f}{\partial y} = 0 + x e^{xz+y} + xy e^{xz+y} \cdot 1 = e^{xz+y}(x + xy)$$

$$\frac{\partial f}{\partial x} = x^2 + xy e^{xz+y} \cdot x = x^2 + x^2 y e^{xz+y}$$

Ableiten

$z = f(x,y)$:



Beispiel: $f(x,y) = x^3 + xy^2$

$$\frac{\partial f}{\partial x} = 3x^2 + y^2, \quad \frac{\partial f}{\partial y} = 0 + x2y = \underline{2xy}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2 + y^2) = \underline{6x}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(3x^2 + y^2) = 0 + 2y = \underline{2y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(2xy) = \underline{2y}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(2xy) = \underline{2x}$$

Annenordens derivert av funksjon av n variable

$y = f(x_1, x_2, \dots, x_n)$

$$\underbrace{\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_j \partial x_i}}_{n^2 \text{ partiell derivert.}} \quad \left(\begin{array}{l}\text{derivere først mhp } x_i \text{ og}\\ \text{deretter mhp. } x_j\end{array}\right)$$

Teorem: Dersom $\frac{\partial^2 f}{\partial x_i \partial x_j}$ og $\frac{\partial^2 f}{\partial x_j \partial x_i}$ er kontinuerlige, så er

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

"blandede partiellderivert er like".

Eksempel $f(x, y, z) = xz + x^2yz$

Finn alle annenordens partiellderivert $(3^2 = 9)$

Finner førstenordens partiellderivert:

$$\frac{\partial f}{\partial x} = z + 2xyz \quad \leftarrow$$

$$\frac{\partial f}{\partial y} = x^2 z$$

$$\frac{\partial f}{\partial z} = x + x^2 y$$

Annenordens:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(z + 2xyz) = 0 + 2yz = 2yz \quad 1$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(z + 2xyz) = 0 + 2xz = 2xz = \frac{\partial^2 f}{\partial x \partial y} \quad 2+3$$

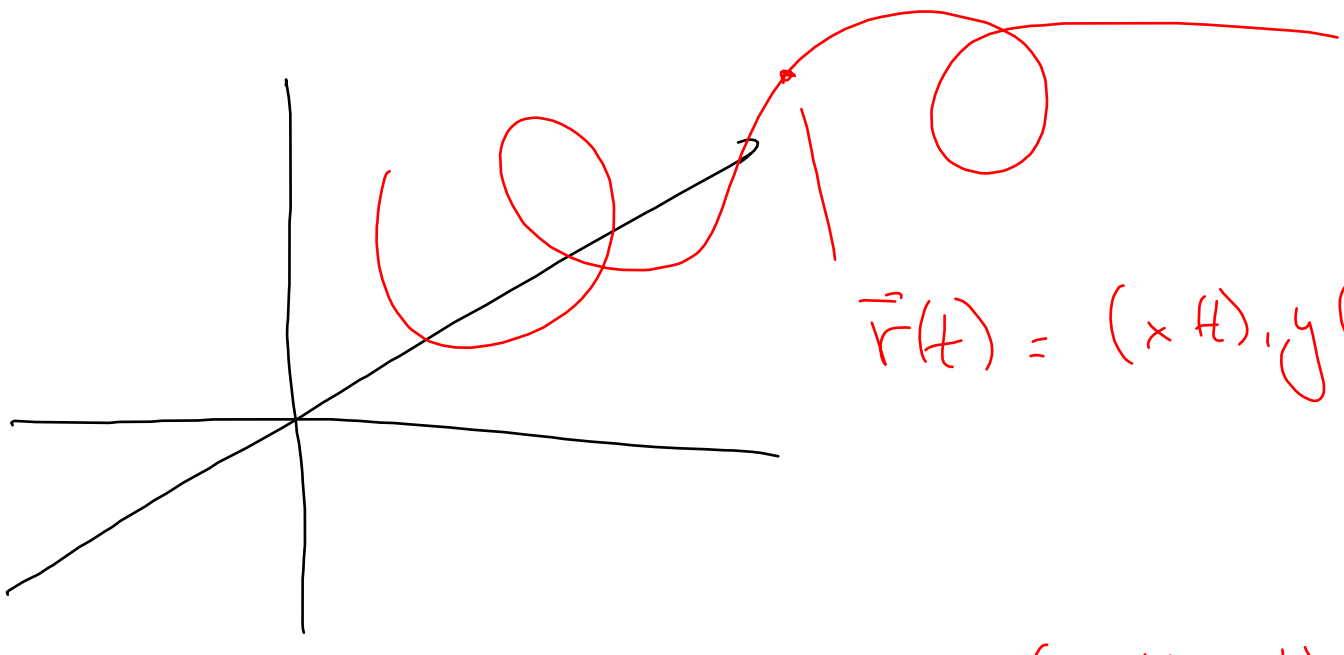
$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}(z + 2xyz) = 1 + 2xy = \frac{\partial^2 f}{\partial x \partial z} \quad 4+5$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2xz$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(x^2 z) = 0 \quad 6$$

$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial z}(x^2 z) = x^2 = \frac{\partial^2 f}{\partial y \partial z} \quad 7+8$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial z}(x + x^2 y) = 0 \quad 9$$



$\vec{r}(t) = (x(t), y(t), z(t))$

n dimensjoner: $\vec{r}(t) = (x_1(t), x_2(t), \dots, x_n(t))$

parametrisert kurve i $\mathbb{R}^n$

Eksempel: $T(x,y,z)$ er temperaturen i punktet (x,y,z)

Fluens posisjon ved tiden t er $\vec{r}(t) = (x(t), y(t), z(t))$

Temperaturen i fluens posisjon ved tiden t:

$f(t) = T(x(t), y(t), z(t))$

Hvor fort endrer temperaturen seg? $f'(t)$, så klart!

Hvordan regner vi ut den, da?

$$f'(t) = \frac{\partial T}{\partial x}(\vec{r}(t)) \cdot x'(t) + \frac{\partial T}{\partial y}(\vec{r}(t)) \cdot y'(t) + \frac{\partial T}{\partial z}(\vec{r}(t)) \cdot z'(t)$$

(kjerneregelen i flere variable)

Kjerneregelen i n variable: Anta at $T(x_1, \dots, x_n)$ og $\vec{r}(t) = (x_1(t), x_2(t), \dots, x_n(t))$ er funksjoner med kontinuerlige deriverte. La

$$f(t) = T(\vec{r}(t)) = T(x_1(t), x_2(t), \dots, x_n(t)).$$

Da er

$$f'(t) = \frac{\partial T}{\partial x_1}(\vec{r}(t))\, x_1'(t) + \frac{\partial T}{\partial x_2}(\vec{r}(t))\, x_2'(t) + \dots + \frac{\partial T}{\partial x_n}(\vec{r}(t))\, x_n'(t)$$

Eksempel: $T(x,y) = xy^2$, $\vec{r}(t) = (t, \sin t)$

$x(t) = t$, $y(t) = \sin t$

$$f(t) = T(x(t), y(t))$$

Kjerneregelen:

$$f'(t) = \frac{\partial T}{\partial x}(\vec{r}(t)) \cdot x'(t) + \frac{\partial T}{\partial y}(\vec{r}(t)) \cdot y'(t)$$

$$= \sin^2 t \cdot 1 + 2t \sin t \cos t$$

$$= \sin^2 t + 2t \sin t \cos t.$$

Mellomregning: $\frac{\partial T}{\partial x} = y^2$, $\frac{\partial T}{\partial y} = 2xy$

$x'(t) = 1$, $y'(t) = \cos t$

$\frac{\partial T}{\partial x}(\vec{r}(t)) = \frac{\partial T}{\partial x}(t, \sin t) = \sin^2 t$

$\frac{\partial T}{\partial y}(\vec{r}(t)) = \frac{\partial T}{\partial y}(t, \sin t) = 2t \sin t$

Alternativ fremgangsmåte:

$$f(t) = T(x(t), y(t)) = x(t)\, y(t)^2$$

$$= t \cdot \sin^2(t)$$

$$f'(t) = 1 \cdot \sin^2(t) + t \cdot 2 \sin t \cos t = \sin^2 t + 2t \sin t \cos t$$

Argument for kjerneregelen:

$$f(t) = T(x(t), y(t))$$

$$f'(t) \Leftarrow \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{T(x(t+\Delta t), y(t+\Delta t)) - T(x(t), y(t))}{\Delta t}$$

$$= \frac{1}{\Delta t}\Big[T(x(t+\Delta t), y(t+\Delta t)) - T(x(t), y(t+\Delta t)) + T(x(t), y(t+\Delta t)) - T(x(t), y(t)) \Big]$$

$$\simeq \frac{\partial T}{\partial x}(\vec{r}(t+\Delta t)) \big(x(t+\Delta t) - x(t)\big) + \frac{\partial T}{\partial y}(\vec{r}(t)) \big(y(t+\Delta t) - y(t)\big)$$

$$\simeq \frac{\partial T}{\partial x}(\vec{r}(t+\Delta t)) \frac{x(t+\Delta t) - x(t)}{\Delta t} + \frac{\partial T}{\partial y}(\vec{r}(t)) \frac{y(t+\Delta t) - y(t)}{\Delta t}$$

$$\downarrow$$

$$\frac{\partial T}{\partial x}(\vec{r}(t)) x'(t) + \frac{\partial T}{\partial y}(\vec{r}(t)) y'(t)$$