Integrasjon

To grunnleggende:

Delvis integrasjon:

$$\int uv'\,dx = uv - \int u v'\,dx$$

Skifte av variabel:

Enkel: $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$ (u, du)

$u = g(x)$

$du = g'(x)\,dx$

Avansert: $\int f(g(x))\,dx$

$u = g(x) \Rightarrow x = h(u)$

$dx = h'(u)\,du$

$$= \int f(u)\,h'(u)\,du$$

Eksempel: $\int x^2 \sin x\,dx$

$u = x^2$ $\quad v' = \sin x$

$u' = 2x$ $\quad v = -\cos x$

$$= x^2(-\cos x) - \int 2x(-\cos x)\,dx = -x^2\cos x + \int 2x\cos x\,dx$$

$u = 2x$ $\quad v' = \cos x$

$u' = 2$ $\quad v = \sin x$

$$= -x^2\cos x + 2x\sin x - \int 2\sin x\,dx$$

$$= -x^2\cos x + 2x\sin x - 2(-\cos x) + C$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C$$

Eksempel $\int x^2 \ln x\,dx$

$u = \ln x$ $\quad v' = x^2$

$u' = \frac{1}{x}$ $\quad v = \frac{x^3}{3}$

$$= \frac{x^3}{3}\ln x - \int \frac{1}{x}\cdot\frac{x^3}{3}\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx = \frac{x^3}{3}\ln x - \frac{1}{3}\cdot\frac{x^3}{3} + C$$

$$= \frac{x^3}{3}\ln - \frac{x^3}{9} + C$$

Eksempel: $I = \int_0^4 e^{\sqrt{x}}\,dx$

$u = \sqrt{x} \Rightarrow x = u^2$

$dx = 2u\,du$

$x = 0 \Rightarrow u = \sqrt{0} = 0$

$x = 4 \Rightarrow u = \sqrt{4} = 2$

$$= \int_0^2 e^u\,2u\,du$$

$u = 2u$ $\quad v' = e^u$

$u' = 2$ $\quad v = e^u$

$$= \left[2ue^u\right]_0^2 - \int_0^2 2e^u\,du$$

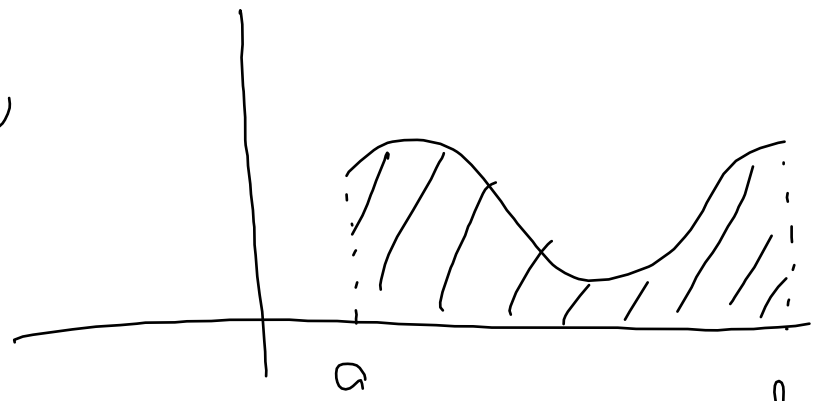
$$= 4e^2 - 0 - \left[2e^u\right]_0^2 = 4e^2 - 0 - [2e^2 - 2e^0]$$

($e^0 = 1$)

$$= \underline{2e^2 + 2}$$

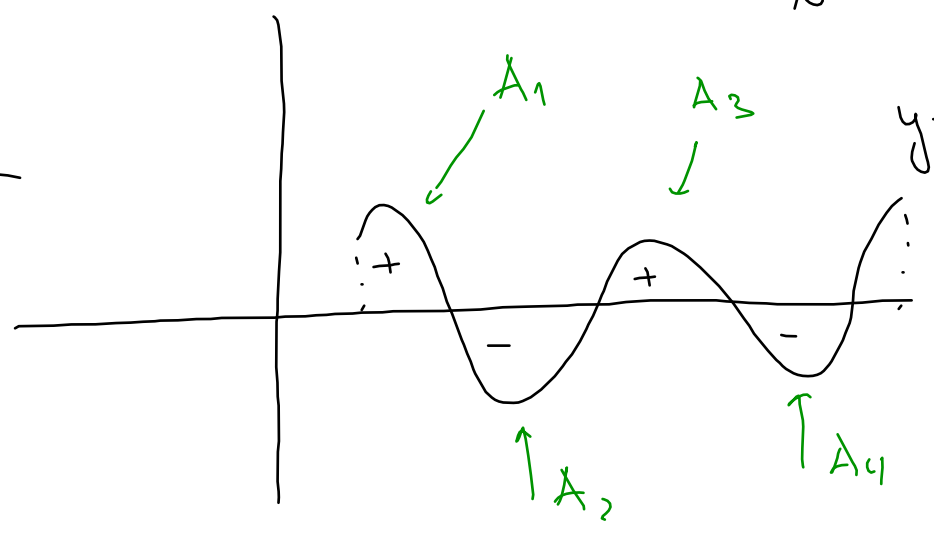
Arealberegning

$f(x) \geq 0$



$$\int_a^b f(x)\,dx = \text{arealet under grafen til } f \text{ fra } a \text{ til } b$$

Generelt

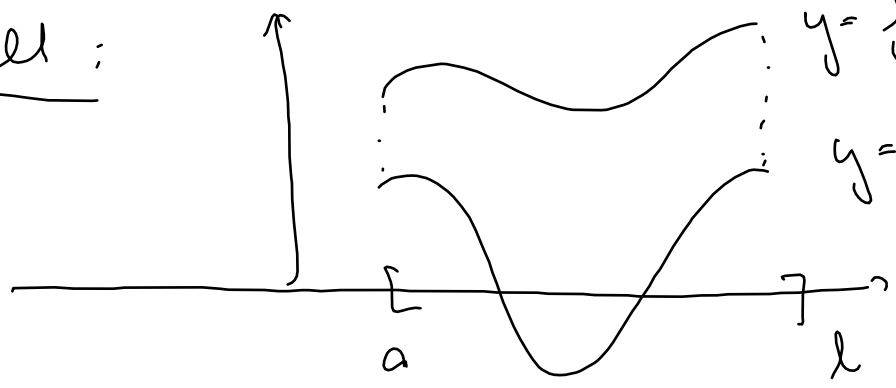


$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3 - A_4.$$

Eksempel

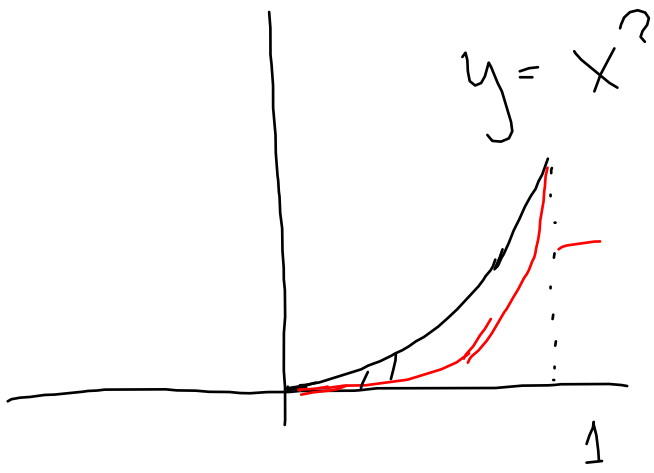
Regn ut arealet mellom funksjonsgrafene $y = x^2$ og $y = x^3$ over intervallet $[0, 1]$.

Generelt:



$$\int_a^b [f(x) - g(x)]\,dx$$

forutsatt at $f(x) \geq g(x)$ for alle $x \in [a, b]$



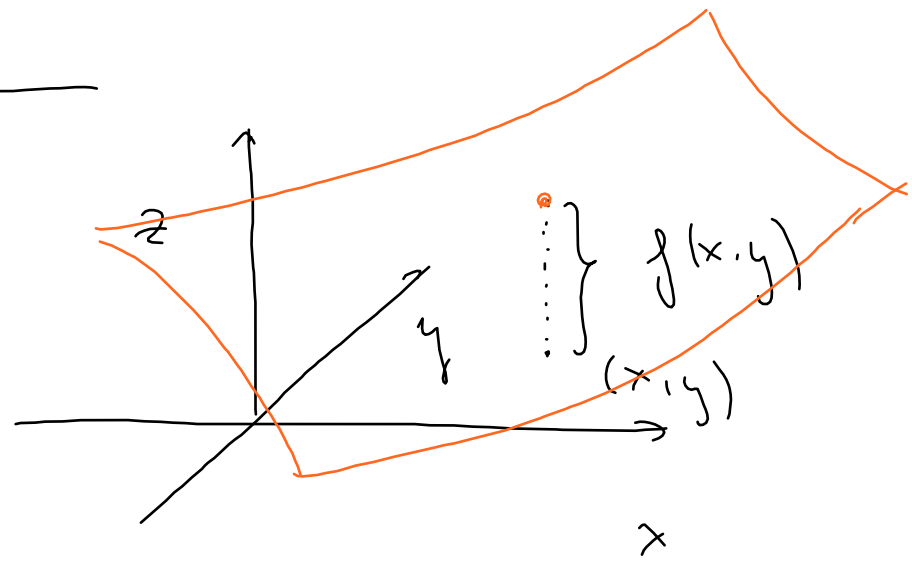
$$A = \int_0^1 (x^2 - x^3)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} - [0 - 0]$$

$$= \frac{4}{12} - \frac{3}{12} = \underline{\underline{\frac{1}{12}}}$$

To variable : $z = f(x,y)$

Flere variable : $z = f(x_1, x_2, \dots, x_n)$



Partiell deriverte :

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \dots, x_i + h, \dots x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

Funksjoner av fle

= derivere mhp x_i når vi later som om alle andre variable er konstante.

Eksempel: $f(x,y,z) = yz\, e^{xyz^2}$

$$\frac{\partial f}{\partial x} = yz\, e^{xyz^2} \cdot yz^2 = y^2 z^3 e^{xyz^2}$$

$$\frac{\partial f}{\partial y} = z e^{xyz^2} + yz\, e^{xyz^2} xz = e^{xyz^2}(z + xyz^3)$$

$$\frac{\partial f}{\partial z} = y e^{xyz^2} + yz \cdot e^{xyz^2} \cdot xy \cdot 2z = e^{xyz^2}(y + 2xy^2z^2)$$

Maks/min-problemer i flere variable

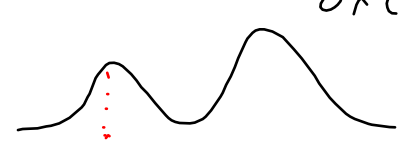
$f(x_1, \dots, x_n)$

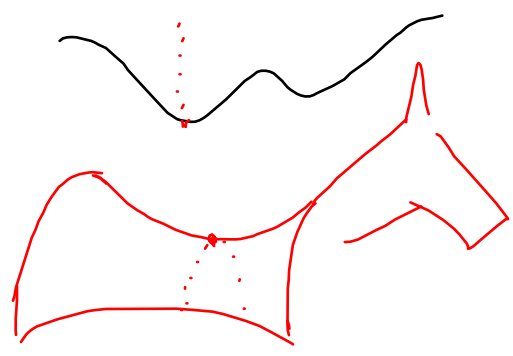
Et stasjonært punkt er et punkt der $\frac{\partial f}{\partial x_i} = 0$ for alle i.

Klassifikasjon: lokalt maksimum

lokalt minimum

sadelpunkter





Annenderivattesten: Hesse-determinanten er

$$Hf(x,y) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{vmatrix} = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

. . . . . .

Anta at $\bar{a}$ er et stasjonært punkt.

(i) Dersom $Hf(\bar{a}) < 0$, så er $\bar{a}$ et sadelpunkt

(ii) Dersom $Hf(a) > 0$, så er $\bar{a}$ enten et lokalt maks eller lokalt minimum: Hvis $\frac{\partial^2 f}{\partial x^2}(\bar{a}) > 0$, så er $\bar{a}$ et lokalt min.

Hvis $\frac{\partial^2 f}{\partial x^2}(a) < a$, så er $\bar{a}$ et lokalt maks.

(iii) Dersom $Hf(a) = 0$, så gir testen ingen konklusjon.

Eksempel: $f(x,y) = xy^2 - 4xy + 2x^2$

Stasjonære punkter:

$$\frac{\partial f}{\partial x} = y^2 - 4y + 4x$$

$$\frac{\partial f}{\partial y} = 2xy - 4x = 2x(y-2)$$

Må løse ligningene

$$\begin{cases} y^2 - 4y + 4x = 0 \\ 2x(y-2) = 0 \end{cases} \quad \begin{array}{l} \nearrow x = 0 \\ \searrow y = 2 \end{array}$$

Anta $x = 0$: $y^2 - 4y + 4 \cdot 0 = 0 \Rightarrow y(y-4) = 0$ $\begin{array}{l} \nearrow y = 0 \\ \searrow y = 4 \end{array}$

Stasjonære punkter: $(0,0), (0,4)$

Anta $y = 2$: $2^2 - 4 \cdot 2 + 4 \cdot x = 0 \Leftrightarrow \underbrace{4 - 8}_{-4} + 4x = 0$

$\Leftrightarrow 4x = 4 \Rightarrow \underline{x = 1}$

Stasjonært punkt: $(1,2)$

Alle $(0,0), (0,4), (1,2)$

Klassifisering: Annenderivate:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(y^2 - 4y + 4x) = 4$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(2xy - 4x) = 2y - 4$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(2xy - 4x) = 2x$$

Hesse determinanten: $Hf(x,y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$

$$4 \cdot 2x - (2y-4)^2 = \underline{8x - 4(y-2)^2}$$

$= -16$ "

Sjekker de stasjonære punktene:

$(0,0)$: $Hf(0,0) = 8 \cdot 0 - 4(0-2)^2 < 0$ sadelpunkt.

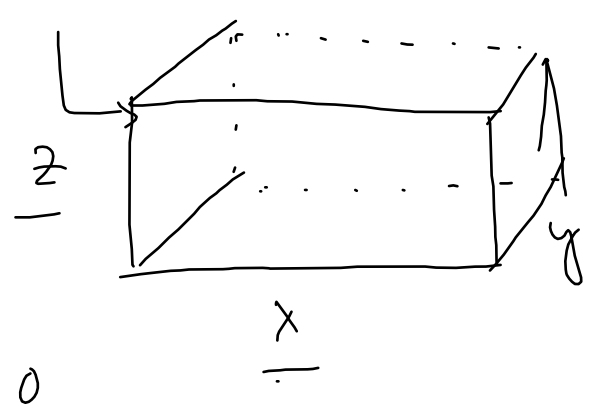
$(0,4)$: $Hf(0,4) = 8 \cdot 0 - 4(4-2)^2 < 0$ sadelpunkt.

$(1,2)$ $Hf(1,2) = 8 \cdot 1 - 4(2-2)^2 = 8 > 0$ / lok. max
Eller
lokal min.

Siden $\frac{\partial^2 f}{\partial x^2}(1,2) = 4 > 0$, da lokalt minimum.

Uoppstilte maks/min i flere variabel:

Eksempel:



Ingen lapp, ingen kvadrede

Bunn fire ganger så dyr som resten.

Størrelse $V = 1000\ cm^3$

Lag den så billig som mulig.

$xyz = 1000$

$z = \frac{1000}{xy}$

k pris per cm^2 på legging

$4k$ — " — i bunnen.

$$P = 4kxy + 1kxz + 2kyz = k\left[4xy + x\frac{1000}{xy} + 2y\frac{1000}{xy}\right]$$

$$= k\left[\underbrace{4xy + \frac{1000}{y} + \frac{2000}{x}}_{f(x,y)}\right]$$

$$\frac{\partial f}{\partial x} = 4y - \frac{2000}{x^2} \qquad \frac{\partial f}{\partial y} = 4x - \frac{1000}{y^2}$$

Setter $4y - \frac{2000}{x^2} = 0 \Rightarrow 4y = \frac{2000}{x^2} \Rightarrow 4yx^2 = 2000$

$4x - \frac{1000}{y^2} = 0 \Rightarrow 4x = \frac{1000}{y^2} \Rightarrow 4xy^2 = 1000$

Tricks:

$$\frac{4yx^2}{4xy^2} = \frac{2000}{1000} = \frac{2}{1} \Rightarrow \frac{x}{y} = \frac{2}{1} \Rightarrow x = 2y$$

Setter inn i $4xy^2 = 1000$

$8y^3 = 1000 \Rightarrow y^3 = \frac{1000}{8} = \frac{10^3}{2^3}$

$y = \frac{10}{2} = \underline{5}$

$x = 2 \cdot 5 = \underline{10}$

$z = \frac{1000}{x \cdot y} = \frac{1000}{50} = \underline{20}$

Billigst: $x = 10, y = 5, z = 20$

