Substitusjon

Observasjon: Hvis F er en antiderivert til f, så

$$\int f(g(x))\,\underbrace{g'(x)}_{\text{magisk faktor}}\,dx = F(g(x)) + C$$

Hvorfor: Hva skulle $F(g(x))$ er en antiderivert til $f(g(x))\,g'(x)$. Ved kjenneregelen:

$$\big(F(g(x))\big)' = F'(g(x))\,g'(x) = f(g(x))\,g'(x) \quad \text{HURRA!}$$

Eksempel: $\int \underbrace{2x}_{g'(x)}\,\underbrace{\sin(x^2)}_{g(x)}\,dx$

$f(x) = \sin(x)$ $\quad F(x) = -\cos x$

$g(x) = x^2$

$g'(x) = 2x$

$= -\cos(x^2) + C$

Sjekk: $\big(-\cos(x^2)\big)' = \sin(x^2)\cdot 2x$

Automatisert metode: $\int f(g(x))\,g'(x)\,dx$

$u = g(x)$

$u' = \dfrac{du}{dx} = g'(x)$

$du = g'(x)\,dx$

$$= \int f(u)\,du = F(u) + C = F(g(x)) + C$$

Eksempel: $\int x e^{-x^2}\,dx$

$u = -x^2$

$\dfrac{du}{dx} = -2x \Rightarrow du = -2x\,dx$

$\Rightarrow dx = \dfrac{du}{-2x}$

$$= \int x e^{u}\,\frac{du}{-2x} = -\frac{1}{2}\int e^u\,du$$

$$= -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2}$$

Eksempel: $\int \dfrac{1}{4x + 2\sqrt{x}}\,dx$

] Desparasjon prøver vi

$u = \sqrt{x} \qquad u^2 = x$

$du = \dfrac{1}{2\sqrt{x}}\,dx$

$dx = 2\sqrt{x}\,du = 2u\,du$

$$= \int \frac{1}{4u^2 + 2u}\,2u\,du$$

$$= \int \frac{1}{2u + 1}\,du$$

$$= \int \frac{1}{v}\,\frac{1}{2}\,dv$$

$$= \frac{1}{2}\int \frac{1}{v}\,dv = \frac{1}{2}\ln|v| + C$$

$$= \frac{1}{2}\ln|2u+1| + C = \frac{1}{2}\ln|2\sqrt{x}+1| + C$$

$\dfrac{2u}{4u^2+2u} = \dfrac{2u}{2u(2u+1)} = \dfrac{1}{2u+1}$

$v = 2u+1$

$\dfrac{dv}{du} = 2 \Rightarrow dv = 2\,du$

$du = \frac{1}{2}\,dv$

Hur hur den magiska faktorn $g'(x)$ saknas?

$$I = \int f(g(x))\,dx$$

$u = g(x) \Rightarrow x = h(u) \Rightarrow \frac{dx}{du} = h'(u) \Rightarrow dx = h'(u)\,du$

$$I = \int f(u)\,h'(u)\,du$$

Exempel: $\int \sin\sqrt{x}\,dx$

$u = \sqrt{x} \Rightarrow x = u^2$

$\frac{dx}{du} = 2u \Rightarrow dx = 2u\,du$

$= \int \sin u \cdot 2u\,du$

Delvis integrering:

$u = 2u \quad v' = \sin u$

$u' = 2 \quad v = -\cos u$

$= -2u\cos u - \int 2(-\cos u)\,du$

$= -2u\cos u + 2\int \cos u\,du$

$= -2u\cos u + 2\sin u + C$

$= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$

Exempel: $I = \int e^{\sqrt{x}}\,dx$

$u = \sqrt{x} \Rightarrow x = u^2$

$\frac{dx}{du} = 2u \Rightarrow dx = 2u\,du$

$= \int e^u\,2u\,du$

Delvis integrering:

$u = 2u \quad v' = e^u$

$u' = 2 \quad v = e^u$

$= 2ue^u - \int 2e^u\,du$

$= 2ue^u - 2e^u + C$

$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$

Hur är formeln

$$\int f(g(x))\,dx = \int f(u)\,h'(u)\,du \quad \text{riktig?}$$

$u = g(x)$, löser ut och får $x = h(u)$. Alltså

$$x = h(u) = h(g(x))$$

Deriverar nu x på högra sidan:

$$1 = h'(g(x))\,g'(x)$$

Därmed (magiska faktorn)

$$\int f(g(x))\,dx = \int f(g(x))\cdot 1\cdot dx = \int f(g(x))\,h'(g(x))\,g'(x)\,dx$$

$$= \int f(u)\,h'(u)\,du$$

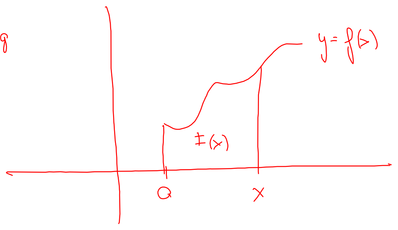
$u = g(x)$

$du = g'(x)\,dx$

Det bestemte integralet

f kontinuerlig

$f \geq 0$

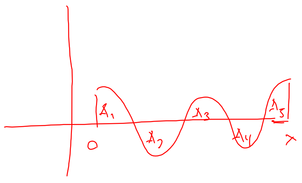


La $F(x)$ være arealet under grafen fra a til x. Vi skal skrive

$$F(x) = \int_a^x f(x)\,dx$$

og kalle dette det bestemte integralet til f fra a til x.

Dersom f kan være negativ, regner vi arealet med forlegn



$$\int_a^x f(x)\,dx = A_1 - A_2 + A_3 - A_4 + A_5$$

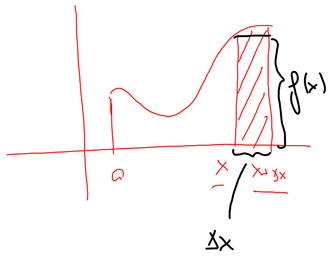
Spørsmål: Hva slags funksjon er $F(x)$?

Vi prøver å derivere den:

$$\frac{F(x+\Delta x) - F(x)}{\Delta x}$$

$$= \frac{\text{areal av stripe}}{\Delta x}$$

$$\approx \frac{f(x)\Delta x}{\Delta x} \longrightarrow f(x)$$



Analysens fundamentalteorem (Del 1): Hvis f er kontinuerlig

så

$$F(x) = \int_a^x f(t)\,dt$$

er antiderivert til f, altså

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Analysens fundamentalteorem (Del 2): Hvis f er kontinuerlig

og G er en antiderivert til f, så

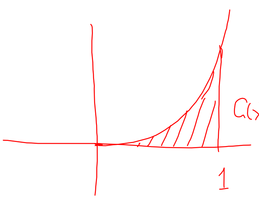
$$\int_a^x f(t)\,dt = G(x) - G(a)$$

Hvorfor? Vi vet at $F(x) = \int_a^x f(t)\,dt$ er en antiderivert til f, og følgelig er $G(x) = F(x) + C$. Altså er

$$G(x) - G(a) = (F(x)+C) - (F(a)+C) = F(x) - F(a)$$

$$= F(x) = \int_a^x f(t)\,dt$$

Eksempel: Regn ut arealet under kurven $f(x) = x^2$ fra $x = 0$ til $x = 1$.



Finn en antiderivert til $f(x) = x^2$

$$G(x) = \int x^2\,dx = \frac{x^3}{3} + C = \frac{x^3}{3}$$

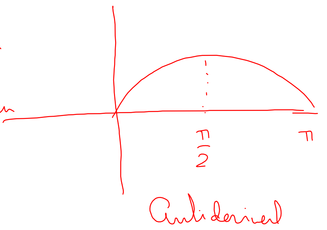
velger en antiderivert

$$\text{Areal} = G(1) - G(0) = \frac{1^3}{3} - \frac{0^3}{3}, \quad C = 0$$

$$= \frac{1}{3}$$

Eksempel: $f(x) = \sin x$

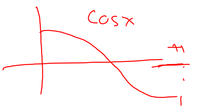
Finn arealet under grafen fra 0 til π:



$$A = \int_0^{\pi} \sin x \, dx$$

Antiderivert

$$G(x) = -\cos x$$



$$= -\cos(\pi) - (-\cos 0) =$$

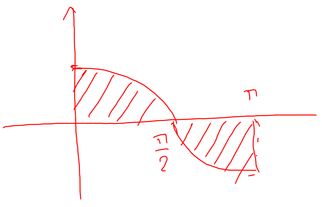
$$= -(-1) - (-1) = 1 + 1 = 2$$

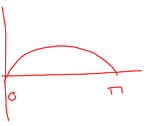
Skrivemåte:

$$\int_0^{\pi} \sin x \, dx = \left[-\cos x \right]_0^{\pi} = -\cos\pi - (-\cos 0)$$

Eksempel:

$$\int_0^{\pi} \cos x \, dx = \left[\sin x \right]_0^{\pi} = \sin\pi - \sin 0 = 0.$$





Konvensjoner: Hvis $b < a$, hva skal da

$$\int_a^b f(x)\,dx \quad \text{bety?}$$

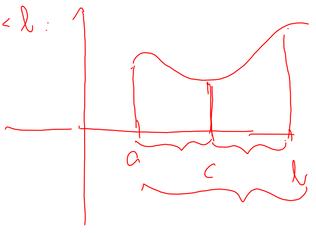
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0.$$

Regneregler:

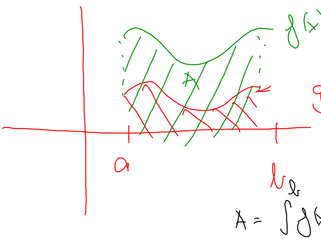
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Hvis $a < c < b$:



$$\int_a^b [c f(x) + d g(x)]\,dx = c\int_a^b f(x)\,dx + d\int_a^b g(x)\,dx$$

Dersom vi har to kontinuerlige funksjoner $f(x)$, $g(x)$, der $f(x) \geq g(x)$, så er arealet mellom $f(x)$ og $g(x)$ over intervallet $[a,b]$ gitt ved

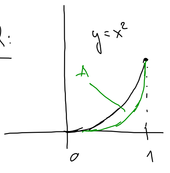


$$A = \int_a^b [f(x) - g(x)]\,dx$$

Hvorfor? Anta $g(x) \geq 0$.

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$$

Eksempel:



$f(x) = x^2$

$g(x) = x^3$

$$A = \int_0^1 (x^2 - x^3)\,dx$$

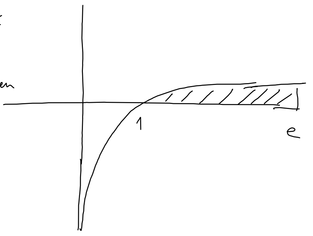
$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \left[\left(\frac{1^3}{3} - \frac{1^4}{4}\right) - \left(\frac{0^3}{3} - \frac{0^4}{4}\right)\right]$$

$$= \frac{1}{3} - \frac{1}{4} - 0 = \frac{4}{12} - \frac{3}{12} = \frac{1}{12}$$

Eksempel: $f(x) = \ln x$

Areal under funksjonsgrafen fra 1 til e.



$$A = \int_1^e \ln(x)\,dx$$

$$= \left[x \ln x - x\right]_1^e$$

$$= \left[(e \underbrace{\ln e}_{1} - e) - (1 \underbrace{\ln 1}_{0} - 1)\right]$$

$$= 1$$

Mellomregning: Løser det ubestemte integralet

$u = \ln x \quad v' = 1$

$u' = \frac{1}{x} \quad v = x$

$$\int \ln x\,dx = \int 1 \cdot \ln x\,dx$$

$$= x \ln x - \int \frac{1}{x} \cdot x\,dx = x \ln x - \int 1\,dx$$

$$= x \ln x - x + C$$