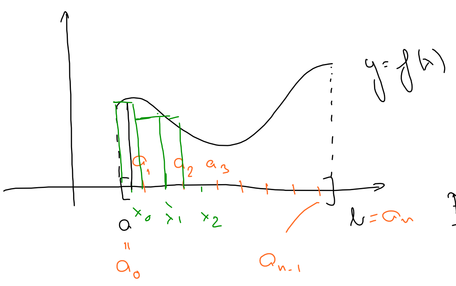
Riemann-summer



Ønsker å regne ut (kanskje bare tilnærmet)

$$\int_a^b f(x)\,dx$$

Partisjon: $\{a_0, a_1, \dots, a_n\}$

Utplukk: $\{x_0, x_1, \dots, x_n\}$

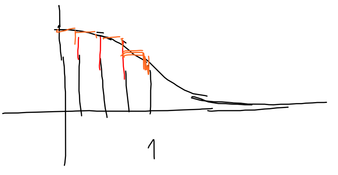
$$R_n = f(x_0)(a_1 - a_0) + f(x_1)(a_2 - a_1) + \dots + f(x_i)(x_{i+1} - x_i) + \dots$$

$$= \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i) \leftarrow \text{Riemann-sum}$$

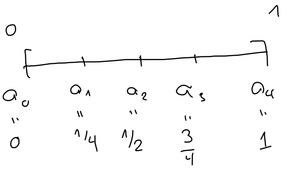
Konvergens: $R_n \to \int_a^b f(x)\,dx$

Eksempel

$$\int_0^1 e^{-x^2}\,dx$$



Finner Riemann-summen R_4 når vi deler intervallet inn i 4 like store intervaller.



$$R_4 = f(x_0)\frac{1}{4} + f(x_1)\cdot\frac{1}{4} + f(x_2)\cdot\frac{1}{4} + f(x_3)\cdot\frac{1}{4}$$

$$= e^{-0^2}\cdot\frac{1}{4} + e^{-\left(\frac{1}{4}\right)^2}\cdot\frac{1}{4} + e^{-\left(\frac{1}{2}\right)^2}\cdot\frac{1}{4} + e^{-\left(\frac{3}{4}\right)^2}\cdot\frac{1}{4}$$

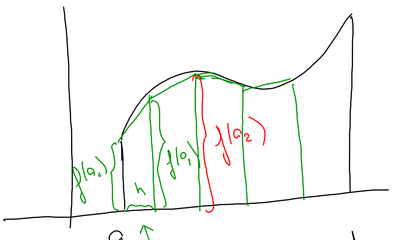
$$= 0.822$$

"Riktig" svar: 0.747

Utplukk venstre endepunkt

$x_0 = 0$, $x_1 = 1/4$, $x_2 = 1/2$, $x_3 = \frac{3}{4}$

TRAPESMETODEN



$\int_a^b f(x)\,dx \approx$ summen av arealet av trapesene

$$\frac{f(a_0)+f(a_1)}{2}\cdot\frac{b-a}{n} + \frac{f(a_1)+f(a_2)}{2}\cdot\frac{b-a}{n} + \frac{f(a_2)+f(a_3)}{2}\cdot\frac{b-a}{n} + \dots$$

$$= \frac{b-a}{2n}\left[f(a_0)+f(a_1)+f(a_1)+f(a_2)+f(a_2)+f(a_3)+\dots+f(a_{n-1})+f(a_n)\right]$$

$$= \frac{b-a}{2n}\left[f(a_0)+2f(a_1)+2f(a_2)+\dots+2f(a_{n-1})+f(a_n)\right]$$

$$= \left[\frac{f(a_0)}{2}+f(a_1)+f(a_2)+\dots+f(a_{n-1})+\frac{f(a_n)}{2}\right]\frac{b-a}{n}$$

"Δx"

Eksempel: $f(x)=e^{-x^2}$, $[0,1]$, $n=4$, $\Delta x=\frac{1}{4}$

$$T_4 = \left[\frac{f(a_0)}{2}+f(a_1)+f(a_2)+f(a_3)+\frac{f(a_4)}{2}\right]\cdot\frac{1}{4}$$

$$= \left[\frac{e^{-0^2}}{2}+e^{-\left(\frac{1}{4}\right)^2}+e^{-\left(\frac{1}{2}\right)^2}+e^{-\left(\frac{3}{4}\right)^2}+\frac{e^{-1^2}}{2}\right]\cdot\frac{1}{4}$$

$$\approx 0.743$$

Fasit: 0.747

Feilestimat: Hvis vi bruker Trapesmetoden med n like lange intervaller, da er feilen

$$\left|T_n - \int_a^b f(x)\,dx\right| \le \frac{K(b-a)^3}{12n^2}$$

der $K \ge |f''(c)|$ for alle $c \in [a,b]$.

Prøver oss på eksempelet: $f(x)=e^{-x^2}$

$$f'(x) = e^{-x^2}(-2x) = -2xe^{-x^2}$$

$$f''(x) = -2e^{-x^2} - (2x)e^{-x^2}(-2x)$$

$$= e^{-x^2}\left(-2+4x^2\right)$$

$$\left|T_4 - \int_0^1 e^{-x^2}\,dx\right| \le \frac{K(b-a)^3}{12n^2} = \frac{2\cdot 1^3}{12\cdot 16} = \frac{1}{6\cdot 16} = \frac{1}{96}$$

$|f''(x)| \le 1\cdot 2 \le 2$

Kan bruke $K=2$ i estimatet.

Vektorer

Et n-tuppel er en endelig rekkefølge

$(a_1, a_2, \dots, a_n)$

av tall.

Eksempel: $(-2, 3, \pi)$ er et 3-tuppel

$(\sqrt{2}, e^2, -17, \frac{3}{14}, 0)$ er et 5-tuppel.

Eksempel: n ansatte, $(t_1, t_2, \dots, t_n)$ arbeidstimer.

Notasjon: $\vec{t} = (t_1, t_2, \dots, t_n)$ (fet t på maskin)

Addisjon: $\vec{t} = (t_1, t_2, \dots, t_n)$, $\vec{s} = (s_1, s_2, \dots, s_n)$

$$\vec{t} + \vec{s} = (t_1 + s_1, t_2 + s_2, \dots, t_n + s_n)$$

$$\vec{t} - \vec{s} = (t_1 - s_1, t_2 - s_2, \dots, t_n - s_n)$$

Eksempel: $\vec{t} = (3, -1, 2, \sqrt{2})$ $\vec{s} = (4, 2, 0, \pi)$

$\vec{t} + \vec{s} = (7, 1, 2, \sqrt{2} + \pi)$

Arbeidstimer: Uke 1 $\vec{t} = (t_1, t_2, \dots, t_n)$

Uke 2 $\vec{s} = (s_1, s_2, \dots, s_n)$

$\vec{t} + \vec{s} = (t_1 + s_1, t_2 + s_2, \dots, t_n + s_n)$

Multiplikasjon med skalar: s, $\vec{t} = (t_1, t_2, \dots, t_n)$

$$s\vec{t} = (st_1, st_2, \dots, st_n)$$

Lønn per time s: $(st_1, st_2, \dots, st_n) = s\vec{t}$

Indreprodukt, skalarprodukt, prikkprodukt:

$\vec{t} = (t_1, t_2, \dots, t_n)$, $\vec{s} = (s_1, s_2, \dots, s_n)$

$\vec{t} \cdot \vec{s} = t_1 s_1 + t_2 s_2 + \dots + t_n s_n$ er et tall!

Eksempel: $\vec{t} = (-2, 1, 4, 3)$, $\vec{s} = (3, -1, -2, 1)$

$\vec{t} \cdot \vec{s} = (-2) \cdot 3 + 1(-1) + 4 \cdot (-2) + 3 \cdot 1$

$= \underbrace{-6 - 1 - 8}_{-15} + 12 = \underline{\underline{-3}}$

Eksempel: Timelønn $\vec{p} = (p_1, p_2, \dots, p_n)$

Arbeidstimer $\vec{t} = (t_1, t_2, \dots, t_n)$

Totale lønnsutgifter: $p_1 t_1 + p_2 t_2 + \dots + p_n t_n = \vec{p} \cdot \vec{t}$

Regneregler:

$$\left.\begin{aligned} \vec{s} + \vec{t} &= \vec{t} + \vec{s} \\ \vec{s} \cdot \vec{t} &= \vec{t} \cdot \vec{s} \end{aligned}\right\} \text{kommutative lover}$$

$\vec{s} \cdot (\vec{t} + \vec{r}) = \vec{s} \cdot \vec{t} + \vec{s} \cdot \vec{r}$ distributive

$a \in \mathbb{R}$ $(a\vec{s}) \cdot \vec{t} = a(\vec{s} \cdot \vec{t})$

OBS: Det er ikke mulig å dele vektorer på hverandre!

Lengden / normen til en vektor: $\vec{t} = (t_1, t_2, \dots, t_n)$

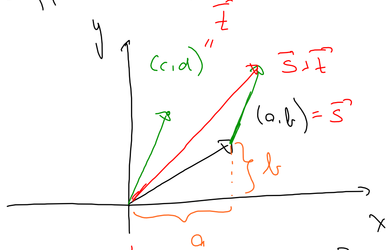
$$\|\vec{t}\| = \sqrt{\overbrace{t_1^2 + t_2^2 + \dots + t_n^2}^{\text{n-tuppel}}} = \sqrt{\vec{t} \cdot \vec{t}}$$

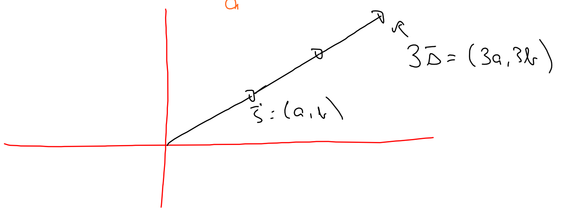
$$\vec{t} \cdot \vec{t} = t_1^2 + t_2^2 + \dots + t_n^2$$

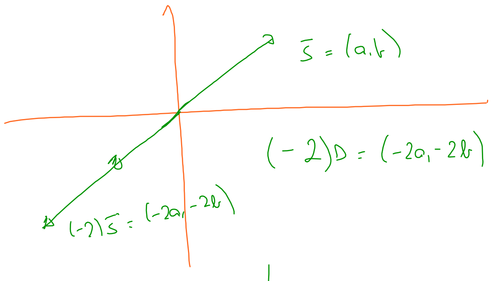
dvs $\|\vec{t}\|^2 = \vec{t} \cdot \vec{t}$

Geometrisk tolkning

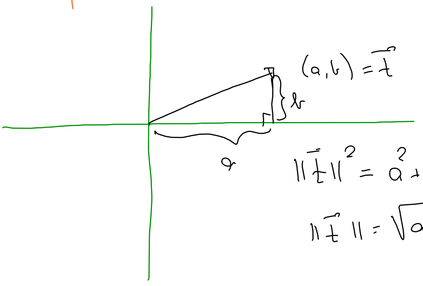
2-tuppel (a,b): (c,d)







Längden:

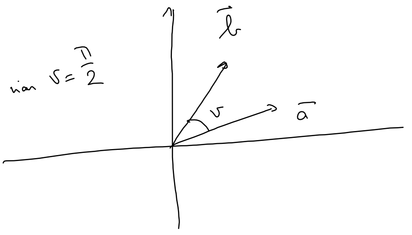


$$\|\vec{t}\|^2 = a^2 + b^2$$

$$\|\vec{t}\| = \sqrt{a^2+b^2}$$

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos v$$

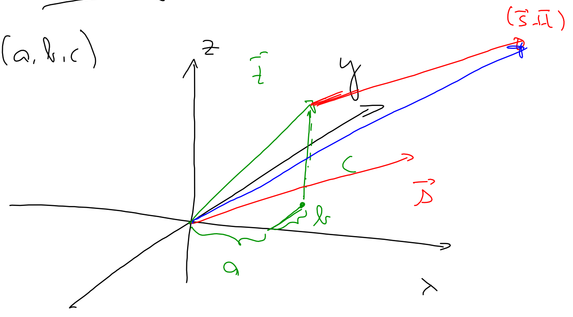
$\vec{a}\cdot\vec{b}=0$ när $\cos v = 0$ dvs när $v=\frac{\pi}{2}$



$$\vec{a}\cdot\vec{b}=0 \iff \vec{a}\perp\vec{b}$$

Talkuing i rammet

$\vec{t} = (a, b, c)$



$\vec{s} = (s_1, s_2, s_3)$, $\vec{t} = (t_1, t_2, t_3)$

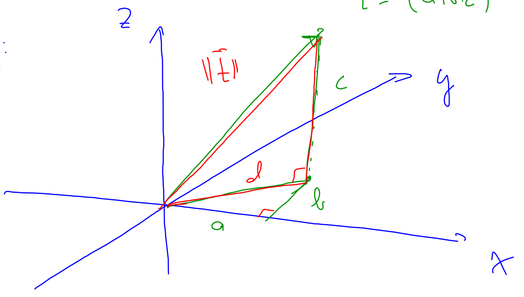
$\vec{s} + \vec{t} = (s_1 + t_1, s_2 + t_2, s_3 + t_3)$

$\vec{s} \cdot \vec{t} = (s_1 t_1, s_2 t, s_3 t_3)$

$\vec{s} \cdot \vec{t} = 0$ betyr at $\vec{s} \perp \vec{t}$.

$t = (a, b, c)$

I enplan:



$d^2 = a^2 + b^2$

$\|t\|^2 = d^2 + c^2 = a^2 + b^2 + c^2$

Hva med n-tupler? Tolkes som en n-dimensjonale vektorer.