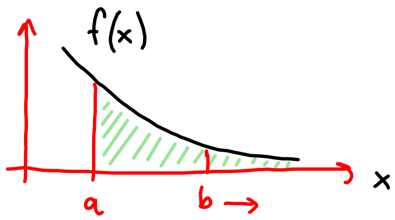
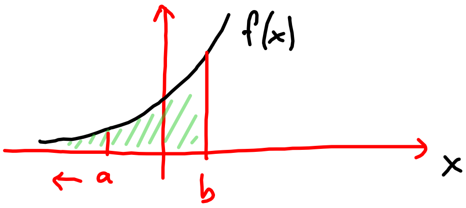
Uegentlige (uekte) integraler (9.5)

$$\int_a^\infty f(x)\,dx \overset{\text{def}}{=} \lim_{b\to\infty} \int_a^b f(x)\,dx$$



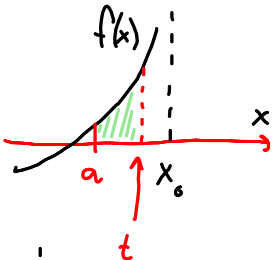
$$\int_{-\infty}^b f(x)\,dx \overset{\text{def}}{=} \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx \overset{\text{def}}{=} \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$$

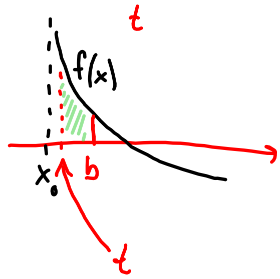


Hvis $f(x)$ har en vertikal asymptote i $x = x_0$, definerer vi

$$\int_a^{x_0} f(x)\,dx \overset{\text{def}}{=} \lim_{t\to x_0^-} \int_a^t f(x)\,dx \quad \text{for } a < x_0$$



$$\int_{x_0}^b f(x)\,dx \overset{\text{def}}{=} \lim_{t\to x_0^+} \int_t^b f(x)\,dx \quad \text{for } b > x_0$$

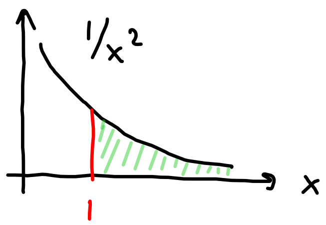


eks. 1

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b$$

$$= \lim_{b\to\infty} \left[-\frac{1}{b} - \left(-\frac{1}{1}\right)\right] = \lim_{b\to\infty} \left[1 - \underbrace{\frac{1}{b}}_{\to 0}\right] = \underline{\underline{1}}$$



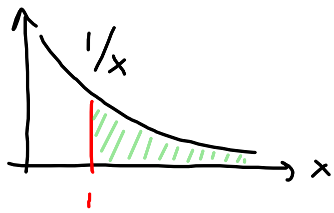
eks. 2

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \left[\ln|x|\right]_1^b$$

$$= \lim_{b\to\infty} \left[\ln b - \overbrace{\ln 1}^{0}\right] = \underline{\underline{+\infty}}$$

(integralet divergerer)



eks. 3

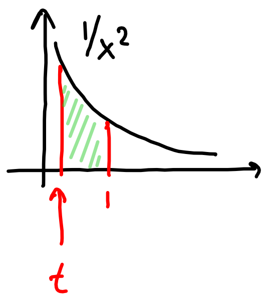
$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}\,dx$$

$$= \lim_{t\to 0^+} \left[-\frac{1}{x}\right]_t^1$$

$$= \lim_{t\to 0^+} \left[-\frac{1}{1} - \left(-\frac{1}{t}\right)\right]$$

$$= \lim_{t\to 0^+} \left[\underbrace{\frac{1}{t}}_{\to +\infty} - 1\right] = \underline{\underline{+\infty}}$$

(integralet divergerer)



Hvis vi får et endelig tall som svar på et uegentlig integral, sier vi at integralet <u>konvergerer</u>. I motsatt fall sier vi at integralet <u>divergerer</u>.

Teorem (p-integralene)

Integralet $\int_1^\infty \frac{1}{x^p}\,dx$ konvergerer for $p > 1$, og divergerer for $p \leq 1$.

Bevis Kan anta $p \neq 1$, siden vi vet fra eks. 2 at det gir divergens. Får:

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left[\frac{1}{-p+1} x^{-p+1}\right]_1^b$$

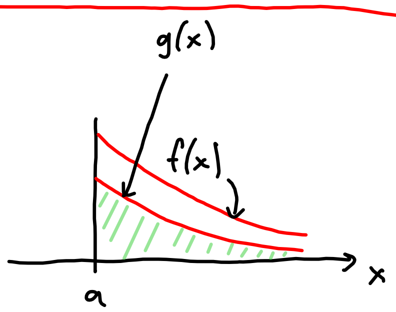
$$= \lim_{b\to\infty}\left[\frac{1}{1-p} b^{1-p} - \frac{1}{1-p}\cdot 1\right]$$

$$= \frac{1}{1-p}\cdot \lim_{b\to\infty}\left[b^{1-p} - 1\right]$$

Her går b^{1-p} mot $+\infty$ hvis $p<1$, og mot 0 hvis $p>1$. $\square$

Teorem (Sammenlikningstesten for integraler)

La f og g være kontinuerlige med $0 \leq g(x) \leq f(x)$ for alle $x \geq A$.



(i) Hvis $\int_a^\infty f(x)\,dx$ konv., så konv. $\int_a^\infty g(x)\,dx$

(ii) Hvis $\int_a^\infty g(x)\,dx$ div., så divergerer $\int_a^\infty f(x)\,dx$

Bevis: Se bok. $\square$

eks. $\int_1^\infty \frac{1}{x^2+\sin^2 x}\,dx$ Konvergerer dette?

Vi bruker sammenlikningstesten, punkt (i): Vi har

$$0 \leq \underbrace{\frac{1}{x^2+\sin^2 x}}_{g(x)} \leq \underbrace{\frac{1}{x^2}}_{f(x)}$$

Vi vet at $\int_1^\infty \frac{1}{x^2}\,dx$ konvergerer (p-integral, $p=2$)

Ergo konvergerer integralet vårt ved sammenlikningstesten. $\square$

Teorem (Grense-sammenlikningstesten for integraler)
Gitt integralene $\int_a^\infty f(x)\,dx$ og $\int_a^\infty g(x)\,dx$, der f og g er positive og kontinuerlige. Hvis

$$L = \lim_{x\to\infty} \frac{f(x)}{g(x)} \quad \text{fins og} \quad 0 < L < \infty$$

så enten konvergerer begge integralene eller divergerer begge integralene.

Bevis Velg konstanter P og Q slik at $0 < P < L < Q$.
Siden $f(x)/g(x) \to L$, har vi for store nok x

$$P < \frac{f(x)}{g(x)} < Q, \quad \text{dvs.} \quad \underline{P\cdot g(x) < f(x) < Q\cdot g(x)}$$

Vi bruker så den vanlige sammenlikningstesten:

$\int f(x)\,dx$ konv $\Rightarrow \int P\cdot g(x)\,dx$ konv, dvs. $\int g(x)\,dx$ konv.

$\int f(x)\,dx$ div. $\Rightarrow \int Q\cdot g(x)\,dx$ div., dvs. $\int g(x)\,dx$ div. $\square$

eks. $\displaystyle\int_1^\infty \frac{x^{3/2}+1}{x^{5/2}+1}\,dx$ Konvergerer dette ?

Funksjonen "går som" $\dfrac{x^{3/2}}{x^{5/2}} = \dfrac{1}{x}$

Vi bruker grense-sammenlikningstesten med $g(x) = \frac{1}{x}$:

$$L = \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\left(\frac{x^{3/2}+1}{x^{5/2}+1}\right)\cdot x}{\left(\frac{1}{x}\right)\cdot x}$$

$$= \lim_{x\to\infty} \frac{(x^{3/2}+1)\cdot x}{x^{5/2}+1} = \lim_{x\to\infty} \frac{x^{5/2}+x}{x^{5/2}+1}$$

Delte på dominerende ledd, kunne også brukt l'Hopital

$$= \lim_{x\to\infty} \frac{1+\frac{x}{x^{5/2}}}{1+\frac{1}{x^{5/2}}} = 1$$

$\frac{x}{x^{5/2}} \to 0$, $\frac{1}{x^{5/2}} \to 0$

Altså diverger integralet vårt ved grense-sammenlikningstesten, fordi vi vet at

$$\int_1^\infty \frac{1}{x}\,dx \text{ divergerer. } \square$$