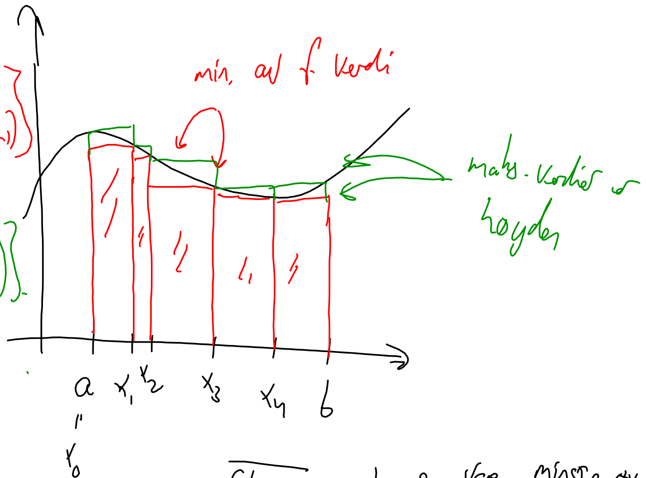
8.2 (def av integralet)

Har 2 definisjoner: def 1 og def 2 (Riemann-summer)

$\Pi = \{a, x_1, x_2, \dots, x_4, b\}$

$$N(\Pi) = \sum_{i=1}^{n} (x_i - x_{i-1}) \min\{f(x_i), f(x_{i-1})\}$$

$$\emptyset(\Pi) = \sum_{i=1}^{4} (x_i - x_{i-1}) \max\{f(x_i), f(x_{i-1})\}$$



Om vi lar Π (partisjon) variere, definerer vi $\overline{\int_a^b} f\,dx$ til å være minste øvre trappesum. Og $\underline{\int_a^b} f\,dx$ er maks. av alle nedre trappesummer.

Vi sier at f er integrerbar om $\underline{\int} f\,dx$ og $\overline{\int} f\,dx$ er samme tall.

§ 2.1 $\quad f(x) = \frac{1}{x}$ på $[1, 2]$

$$\Pi = \left\{ 1, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}, 2 \right\}$$

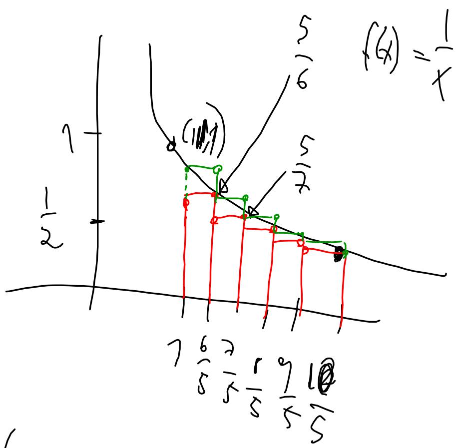
$$\emptyset(\Pi) = \frac{1}{\cancel{5}} \cdot \frac{\cancel{5}}{5} + \frac{1}{\cancel{5}} \cdot \frac{\cancel{5}}{6} + \dots + \frac{1}{\cancel{5}} \cdot \frac{\cancel{5}}{9} \approx 0.746$$

$$N(\Pi) = \frac{1}{\cancel{5}} \cdot \frac{\cancel{5}}{6} + \frac{1}{\cancel{5}} \cdot \frac{\cancel{5}}{7} + \frac{1}{\cancel{5}} \cdot \frac{\cancel{5}}{8} + \dots + \frac{1}{\cancel{5}} \cdot \frac{\cancel{5}}{10}$$

$$= \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \approx 0.646$$

Så $0.646 < \ln 2 < 0.746$

$\ln 2 \approx 0.693$



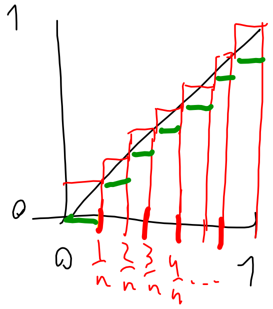
$$\int_1^2 \frac{1}{x} = \ln 2$$

8.2.5

$f(x) = x$

$f : [0,1] \to [0,1]$

$\Pi_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$



Husk $1+2+3+\dots+n = \frac{(n+1)n}{2}$

a) Finn uttrykk for $\text{Ø}(\Pi_n)$ og $N(\Pi_n)$.

$$\text{Ø}(\Pi_n) = \sum_{i=1}^{n} \underbrace{\frac{1}{n} \cdot \frac{i}{n}}_{\text{areal boks nr } i} = \frac{1}{n^2} \sum_{i=1}^{n} i = \frac{1}{n^2}\left(\frac{(n+1)n}{2}\right) = \frac{(n+1)}{2n}$$

$$N(\Pi_n) = \sum_{i=1}^{n} \underbrace{\frac{1}{n} \cdot \frac{i-1}{n}}_{\text{areal i boks nr } i} = \frac{1}{n^2} \overbrace{\sum_{i=1}^{n} i-1}^{1+2+3+\dots+n-1} = \frac{1}{n^2} \frac{n(n-1)}{2} = \frac{n-1}{2n}$$

b) Finn $\overline{\int} f\,dx$ og $\underline{\int} f\,dx$.

fordi $\overline{\int} f\,dx = \inf\{$alle mulige øvre trappe-summer$\}$

Har alltid $\frac{n-1}{2n} = N(\Pi_n) \le \underline{\int} f\,dx \le \overline{\int} f\,dx \le \text{Ø}(\Pi_n) = \frac{n+1}{2n}$

fordi $\underline{\int} f\,dx$ er sup av $N(\Pi_n)$

fordi $N(\Pi_n) \le \text{Ø}(\Pi_n)$

$n \to \infty$ $\qquad \to \frac{1}{2}$

$\frac{1}{2} \le \underline{\int} f\,dx \le \overline{\int} f\,dx \le \frac{1}{2}$

Finere og finere oppdeling av partisjonen gir lik $\overline{\int} f\,dx$ og $\underline{\int} f\,dx$.

$\Rightarrow$ så $\underline{\int} f\,dx = \overline{\int} f\,dx = \frac{1}{2}$ ✓

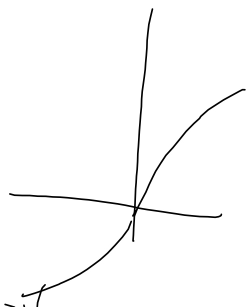
c) f er integrerbar fordi $\underline{\int} f\,dx$ og $\overline{\int} f\,dx$ er like.

Så $\int_0^1 x\,dx = \frac{1}{2}$ fordi $\underline{\int} f\,dx = \frac{1}{2}$.

8.3.4 a) $\int_{-\sqrt{\pi}}^{\sqrt{\pi}} x\cos(x^2)\,dx$

Løsn 1 (funker på b, c, d, e også) er å finne antiderivert F

$$\boxed{\int_a^b f(x) = F(b) - F(a) \quad \text{der } F'(x) = f(x)}$$



$$\frac{1}{2}\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \underbrace{2x}_{u'} \cos(\underbrace{x^2}_{u})\,dx = \frac{1}{2}\Big[+\sin(x^2)\Big]_{-\sqrt{\pi}}^{\sqrt{\pi}}$$

Siden $(\sin x^2)' = 2x\cos x^2$

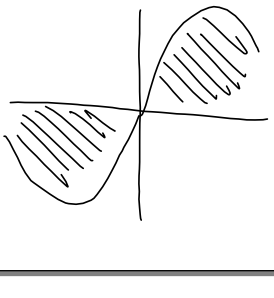
$$= \frac{1}{2}\left(\sin\pi - (\sin(-\pi))\right) = 0.$$

~~odd~~ odd/even function

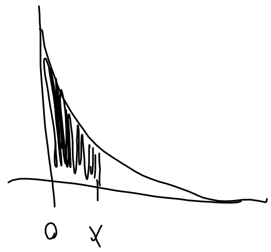
Løsn 2 Merk at hvis $f(x) = x\cos(x^2)$ så er $\boxed{f(-x) = -f(x)}$

Så $\int_{-\sqrt{\pi}}^{\sqrt{\pi}} f(x)\,dx = \int_{-\sqrt{\pi}}^{0} f(x)\,dx + \int_{0}^{\sqrt{\pi}} f(x)\,dx = \int_{0}^{\sqrt{\pi}} -f(x)\,dx + \int_{0}^{\sqrt{\pi}} f(x)\,dx$

$= \underline{0}.$



8.3.7 a) $\lim\limits_{x \to 0} \dfrac{\int_0^x e^{-t^2}\,dt}{x}$



$e^{-t^2} = \dfrac{1}{e^{t^2}}$

Husk: Hvis $F' = f$, så er

$$\int_0^x f(t)\,dt = F(x) - F(0)$$

Deriverer mhp x:

$$\left[\int_0^x f(t)\,dt\right]' = F'(x) = f(x)$$

Så vi vet den deriverte ⇒ L'Hôpital.

$$\lim_{x \to 0} \frac{\int_0^x e^{-t^2}\,dt}{x} \overset{H}{=} \lim_{x \to 0} \frac{e^{-x^2}}{1} = 1$$

$=$

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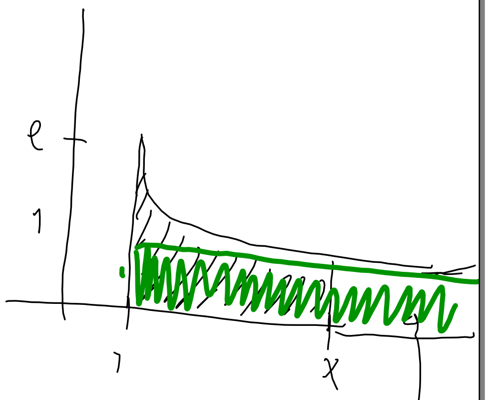
b)

$$A = \lim_{x\to\infty} \frac{\int_1^x e^{\frac{1}{t}}\,dt}{x^2} \qquad \begin{matrix}\to\infty \\ \to\infty\end{matrix}$$



Har $\frac{\infty}{\infty}$ - uttrykk.

$$\int_0^x f(t)\,dt = F(x) - F(0) \quad \text{der } F' = f$$

Så $$\left[\int_0^x f(t)\,dt\right]' = F'(x) = f(x)$$

$$A \overset{H}{=} \lim_{x\to\infty} \frac{e^{\frac{1}{x}}}{2x} = \underline{\underline{0}}$$

$f = e^{\frac{1}{x}}$ er synkende og ≥ 1.

$f \geq 1$

Så $\int_1^x f\,dx \geq \int_1^x 1\,dx = x - 1$

8.3.9 Anta f er kontinuerlig. Vis at det finnes $c \in (a,b)$ $\quad b \neq a$

slik at $\int_a^b f(x)\,dx = f(c)(b-a)$

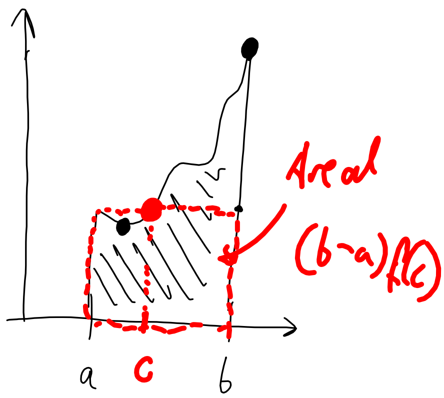
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$= \underbrace{\frac{F(b)-F(a)}{(b-a)}}\,(b-a)$$

middelverdisetningen

$$= F'(c)(b-a)$$

$$= f(c)(b-a).$$



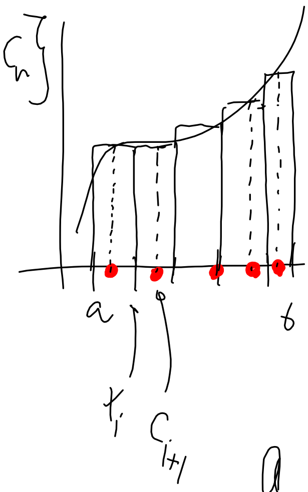
$\boxed{\text{MVS: } \frac{F(b)-F(a)}{b-a} = F'(c)}$

8.5.2 Riemann-summer

• Def 2 av integralet.

$$\Pi = \{ \underset{a}{\overset{\shortparallel}{x_0}}, x_1, \ldots, \underset{b}{\overset{\shortparallel}{x_n}} \}$$

• Velger $c_i \in [x_{i-1}, x_i]$. $\quad U = \{c_1, \ldots, c_n\}$



Da er Riemann-summen:

$$R(\Pi, U) = \sum_{i=1}^{n} \underbrace{(x_i - x_{i-1})}_{\text{bredde boks}} \underbrace{f(c_i)}_{\text{høyde}}$$

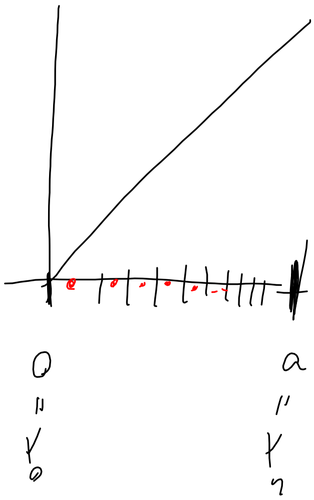
$f(x) = x$ $\quad \Pi_n = \{0, x_1, x_2, \ldots, \underset{a}{\overset{\shortparallel}{x_n}}\}$

$$c_i = \frac{x_i + x_{i-1}}{2}$$

Da er

$$R(\Pi, U) = \sum_{i=1}^{n} (x_i - x_{i-1}) \frac{x_i + x_{i-1}}{2}$$

$$= \frac{1}{2} \sum_{i=1}^{n} x_i^2 - x_{i-1}^2 = \frac{1}{2} \Big((x_1^2 - x_0^2) + (x_2^2 - x_1^2) + (x_3^2 - x_2^2) + \ldots + (x_n^2 - x_{n-1}^2) \Big)$$



$$= \frac{1}{2}(x_n^2 - x_0^2) = \frac{1}{2} a^2 \quad ✓$$