9.2.1 f) $\int \sin(\sqrt[3]{x})\,dx$

$u = \sqrt[3]{x}$ $\quad x = u^3$

$dx = 3u^2\,du$ Deriverer mhp u.

$= \int \underbrace{3\sin(u)}_{u'} \overset{v}{u^2}\,du$

$= -3\cos u \cdot u^2 + \int 3\cos u \cdot 2u\,du$

$\boxed{\int u'v = uv - \int uv'}$

$= -3\cos u \cdot u^2 + 6\int \cos u \cdot u\,du$

$= -3\cos u \cdot u^2 + 6\left(\sin u \cdot u - \int \sin u\,du\right)$

$= -3\cos u \cdot u^2 + 6(\sin u \cdot u + \cos u) + C$

$= -3\cos u \cdot u^2 + 6\sin u \cdot u + 6\cos u + C$ $\quad$ Husk $u = \sqrt[3]{x}$

$= -3\cos\sqrt[3]{x} \cdot \sqrt[3]{x^2} + 6\sin\sqrt[3]{x} \cdot \sqrt[3]{x} + 6\cos\sqrt[3]{x} + C$ ✓

nov 18-08:15

9.2.3 d) $\int_0^3 \arctan\sqrt{x}\, dx$

$u = \sqrt{x} \rightsquigarrow x = u^2$

så $dx = 2u \cdot du$

delvis-integrasjon

$u \quad v'$

$$= \int_0^{\sqrt{3}} \arctan u \cdot 2u\, du$$

$$= \underbrace{\Big[u^2 \cdot \arctan u\Big]_0^{\sqrt{3}}} - \int_0^{\sqrt{3}} u^2 \cdot \frac{1}{1+u^2}\, du$$

$$= 3 \cdot \frac{\pi}{3} - \int_0^{\sqrt{3}} \frac{u^2}{1+u^2}\, du$$

$$= \pi - \int_0^{\sqrt{3}} 1 - \frac{1}{1+u^2}\, du$$

$$= \pi - \Big[u - \arctan u\Big]_0^{\sqrt{3}} = \pi - \left(\sqrt{3} - \frac{\pi}{3}\right) = \underline{\underline{\frac{4\pi}{3} - \sqrt{3}}}$$

Hvis grensene førstnevnt er a og b så er de nye $u(a)$ og $u(b)$

$\arctan(\sqrt{3}) = \frac{\pi}{3}$

$$\frac{u^2}{1+u^2} = \frac{u^2+1-1}{1+u^2} = 1 - \frac{1}{1+u^2}$$

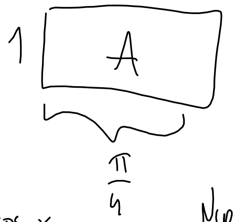
nov 18-08:24

9.225

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx \qquad n = 0, 1, 2, \dots$$

a) Regn ut I_0 og I_1.

$$I_0 = \int_0^{\frac{\pi}{4}} \tan^0 x \, dx = \int_0^{\frac{\pi}{4}} 1 \, dx = \frac{\pi}{4}$$



$$I_1 = \int_0^{\frac{\pi}{4}} \tan \, dx = -\int_0^{\frac{\pi}{4}} \frac{-\sin x}{\cos x} \, dx$$

$u = \cos x$

$du = -\sin x \, dx$

Nye grenser blir

$u(0) = 1$

$u\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$$= -\int_1^{\sqrt{2}/2} \frac{1}{u} \, du = -\left[\ln u\right]_1^{\sqrt{2}/2}$$

$$= -\left(\ln \frac{\sqrt{2}}{2}\right) = -\left(\ln \sqrt{2} - \ln 2\right)$$

$$= -\left(\tfrac{1}{2}\ln 2 - \ln 2\right) = \underline{\tfrac{1}{2}\ln 2}$$

$\boxed{\ln(a^b) = b \ln a}$

B) Vis at $\tan^{n+2} x = \tan^n x \left(\frac{1}{\cos^2 x} - 1 \right)$ (✱)

og bruk dette til å vise at

$$I_{n+2} = \frac{1}{n+1} - I_n \quad (✱✱)$$

Viser (✱). $\tan^{n+2} x = \tan^n x \tan^2 x.$

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x} = \underline{\underline{\frac{1}{\cos^2 x} - 1}}$$

$\boxed{\sin^2 x + \cos^2 x = 1}$

Så da er $\tan^{n+2} x = \tan^n x \left(\frac{1}{\cos^2 x} - 1 \right)$ ✓

(✱✱) er ekvivalent m/ $I_{n+2} + I_n = \frac{1}{n+1}$:

$$\int_a^b f+g = \int_a^b f + \int_a^b g$$

$$I_{n+2} + I_n = \int_0^{\pi/4} \tan^{n+2} \cancel{x}\, dx + \int_0^{\frac{\pi}{4}} \tan^n x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n+2} x + \tan^n x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^n x \left(\tan^2 x + 1 \right) dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^n x \left(\frac{1}{\cos^2 x} - 1 + 1 \right) dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^n x \cdot \underbrace{\frac{1}{\cos^2 x} \, dx}_{du}$$

$u = \tan x$
Nye grenseverdier
blir $u(0) = 0$
og $u(\frac{\pi}{4}) = 1$

$$= \int_0^1 u^n \, du = \underline{\underline{\frac{1}{n+1}}}$$

$\boxed{\text{Husk } (\tan x)' = \frac{1}{\cos^2 x}}$

$\boxed{u = \tan x \quad du = \frac{1}{\cos^2 x} dx}$

nov 18-08:42

c) Vis v/ induksjon at

$$I_{2n+1} = \frac{(-1)^n}{2}\left[\ln 2 - \underbrace{\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{n+1}}{n}\right)}_{\sum_{i=1}^{n} \frac{(-1)^{i+1}}{i}}\right]$$

for $n = 1, 2, \dots$

Så kan også skrive

$$I_{2n+1} = \frac{(-1)^n}{2}\left[\ln 2 - \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i}\right]$$

$n = 1, 2, \dots$

Husk Induksjon:
- Vis først $n=1$
- Anta så at $n=k$ stemmer. Bruk dette til å utlede at det stemmer for $k+1$. ✓

$n=1$?

$$I_{2\cdot1+1} = I_3 \overset{(B)}{=} \frac{1}{2} - I_1$$
$$= \frac{1}{2} - \frac{1}{2}\ln 2$$

og formelen for $n=1$ sier $\frac{-1}{2}\left[\ln 2 - \frac{1}{1}\right] = -\frac{1}{2}\ln 2 + 1$

Samme svar! Så formel stemmer for $n=1$.

$n=1$

$$I_3 = \frac{-1}{2}\left[\ln 2 - \frac{(-1)^2}{1}\right]$$
$$= \frac{-1}{2}\left[\ln 2 - 1\right]$$
$$= \frac{1}{2}(1 - \ln 2)$$

$n=2$

$$I_5 = \frac{1}{2}\left(\ln 2 - \left(\frac{1}{1} - \frac{1}{2}\right)\right)$$
$$= \frac{1}{2}\left(\ln 2 - 1 + \frac{1}{2}\right) \dots$$
$$= \frac{1}{2}\left(\ln 2 - \frac{1}{2}\right) \overset{?}{=} \int_0^{\pi/4} \tan^5 dx$$

nov 18-08:53

Anta formelen

$$I_{2n+1} = \frac{(-1)^n}{2}\left[\ln 2 - \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i}\right]$$ stemmer for $n=k$.

(stemmer for $n=k$ pr antakelse)

Regner ut for $k+1$:

$$\boxed{\text{Ⓑ}\quad I_{n+2} = \frac{1}{n+1} - I_n}$$

$$I_{2(k+1)+1} = I_{2k+3} \overset{\text{Ⓑ}}{=} \frac{1}{2k+2} - I_{2k+1}$$

$$\boxed{I_{2k+3} = I_{\underbrace{2k+1}_{n}+2} \quad \text{så Ⓑ sier} \quad I_{2k+3} = \frac{1}{2k+1} - I_{2k+1}}$$

Bruker ind. hypotese

$$= \frac{1}{2k+2} - \left(\frac{(-1)^k}{2}\left[\ln 2 - \sum_{i=1}^{k}\frac{(-1)^{i+1}}{i}\right]\right)$$

$$= \frac{1}{2}\,\frac{1}{k+1} + \frac{(-1)^{k+1}}{2}\left[\ln 2 - \sum_{i=1}^{k}\frac{(-1)^{i+1}}{i}\right]$$

$$= \frac{1}{2}\left[\frac{1}{k+1} + \frac{(-1)^{k+1}}{1}\left[\ln 2 - \sum_{i=1}^{k}\frac{(-1)^{i+1}}{i}\right]\right]$$

Merk $(-1)^{k+1}\cdot(-1)^{k+1} = 1$

$$= \frac{(-1)^{k+1}}{2}\left[\frac{(-1)^{k+1}}{k+1} + \left[\ln 2 - \sum_{i=1}^{k}\frac{(-1)^{i+1}}{i}\right]\right]$$

$$= \frac{(-1)^{k+1}}{2}\left[\frac{(-1)^{k+1}}{k+1} + \ln 2 - \sum_{i=1}^{k}\frac{(-1)^{i+1}}{i}\right]$$

$$\boxed{\underline{\text{Triks}} \quad \frac{(-1)^{k+1}}{k+1} = \frac{-(-1)^{k+2}}{k+1}}$$

$$= \frac{(-1)^{k+1}}{2}\left[\frac{-(-1)^{k+2}}{k+1} + \ln 2 - \sum_{i=1}^{k}\frac{(-1)^{i+1}}{i}\right]$$

$$= \frac{(-1)^{k+1}}{2}\left[\ln 2 - \sum_{i=1}^{k+1}\frac{(-1)^{i+1}}{i}\right]$$

Fordi $\frac{(-1)^{k+2}}{k+1}$ er sisteleddet i $\sum_{i=1}^{k+1}\frac{(-1)^{i+1}}{i}$

$$= I_{2(k+1)+1} \quad \checkmark$$

Så $\forall$ å anta at I_k stemmer, vise vi at I_{k+1} også stemmer. $\forall$ induksjon stemmer formelen for alle n.

nov 18-09:21

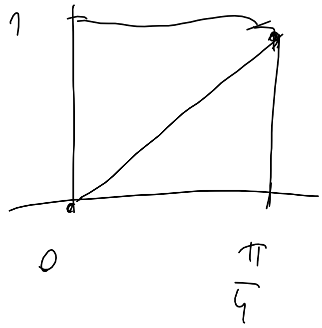
d) Forklar hvorfor $\lim_{n\to\infty} \underbrace{\int_0^{\pi/4} \tan^n x\, dx}_{I_n} = 0$. Bruk dette til å vise at $\ln 2 = \lim_{n\to\infty}\left(1-\frac{1}{2}+\frac{1}{3}-\dots\right)$

Metode 1

$$\lim_{n\to\infty}\int_0^{\frac{\pi}{4}} \tan^n x\, dx \overset{?}{=} \int_0^{\frac{\pi}{4}} \lim_{n\to\infty} \tan^n x\, dx$$

Tror dette er lov.

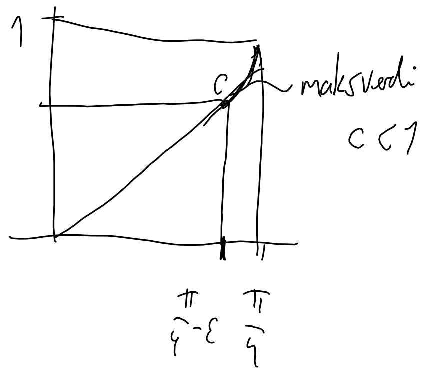
$$= \int_0^{\pi/4} 0\, dx = 0$$



Metode 2 (garantert lov, litt mer teknisk)

På intervallet $[0, \frac{\pi}{4}-\varepsilon]$ er $0 \le \tan x < 1$

$(0<\varepsilon<\frac{\pi}{4})$



Da er

$$\int_0^{\frac{\pi}{4}} \tan^n x\, dx = \int_0^{\frac{\pi}{4}-\varepsilon} \tan^n x\, dx + \int_{\frac{\pi}{4}-\varepsilon}^{\pi/4} \tan^n x\, dx$$

max(tan x) = $\tan(\frac{\pi}{4}-\varepsilon)$ = $c < 1$

$$\le c^n\left(\frac{\pi}{4}-\varepsilon\right) + \varepsilon \to \varepsilon$$

fordi $c^n \to 0$ når $n\to\infty$

Da er $\int_a^b f \le (b-a)\cdot \max f$

Så $\lim_{n\to\infty} I_n \le \varepsilon$ for alle $\varepsilon > 0$. Lar vi $\varepsilon \to 0$ vil vi få $\lim_{n\to\infty} I_n = 0$.

Fra formel er

$$\underbrace{(-1)^n I_{2n+1}}_{0} = \frac{1}{2}\Big[\underbrace{\ln 2 - \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i}}_{=\,0 \text{ når } n\to\infty}\Big]$$

La $n\to\infty$.