LØSNINGSFORSLAG PRØVEEKSAMEN MAT 1100

1 a)

$$\vec{F}(x,y,z) = \begin{pmatrix} x^2\cos z \\ \ln(xy) \\ \tan(yz) \end{pmatrix}$$

$$\vec{F}'(x,y,z) = \begin{pmatrix} \frac{\partial}{\partial x}(x^2\cos z) & \frac{\partial}{\partial y}(x^2\cos z) & \frac{\partial}{\partial z}(x^2\cos z) \\ \frac{\partial}{\partial x}(\ln(xy)) & \frac{\partial}{\partial y}(\ln(xy)) & \frac{\partial}{\partial z}(\ln(xy)) \\ \frac{\partial}{\partial x}(\tan(yz)) & \frac{\partial}{\partial y}(\tan(yz)) & \frac{\partial}{\partial z}(\tan(yz)) \end{pmatrix}$$

$$= \begin{pmatrix} 2x\cos z & 0 & -x^2\sin z \\ \frac{1}{x} & \frac{1}{y} & 0 \\ 0 & \frac{z}{\cos^2(yz)} & \frac{y}{\cos^2(yz)} \end{pmatrix}$$

b)

$$\vec{F}'(1,2,0) = \begin{pmatrix} 2\cos 0 & 0 & -\sin 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{2}{\cos^2 0} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\underline{\vec{F}'(1,2,0)^2} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ \frac{5}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\underline{\vec{F}'(1,2,0)^3} = \begin{pmatrix} 4 & 0 & 0 \\ \frac{5}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{pmatrix} = \underline{\begin{pmatrix} 8 & 0 & 0 \\ \frac{21}{4} & \frac{1}{8} & 0 \\ 0 & 0 & 8 \end{pmatrix}}$$

2.

$$f(x,y) = y\sin(xy) \qquad \vec{a} = \left(\tfrac{\pi}{6}, -1\right) \qquad \vec{r} = (2,3)$$

$$\vec{\nabla} f(x,y) = \left(\frac{\partial}{\partial x} f(x,y), \frac{\partial}{\partial y} f(x,y)\right) = \left(y^2\cos xy, \sin(xy) + xy\cos(xy)\right)$$

$$\underline{f'(\vec{a},\vec{r})} = \vec{\nabla}\left(\tfrac{\pi}{6}, -1\right)\cdot(2,3) = \left(\cos\left(-\tfrac{\pi}{6}\right), \sin\left(-\tfrac{\pi}{6}\right) - \tfrac{\pi}{6}\cos\left(-\tfrac{\pi}{6}\right)\right)\cdot(2,3)$$

$$= \left(\tfrac{1}{2}\sqrt{3}, -\tfrac{1}{2} - \tfrac{\pi}{12}\sqrt{3}\right)(2,3) = \underline{\sqrt{3} - \tfrac{3}{2} - \tfrac{\pi}{4}\sqrt{3}}$$

$\underline{3}$ P er en pyramide med hjørner

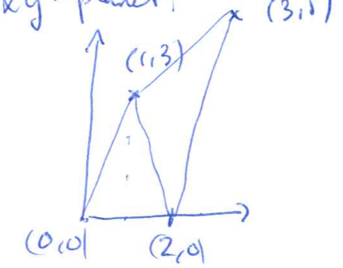
$(0,0,0),\ (2,0,0),\ (1,3,0),\ (3,5,0),\ (1,2,6).$

Volumet til P er summen av volumene ~~ets~~ av pyramidene med hjørner i

$(0,0,0),(2,0,0),(1,3,0),(1,2,6)$ (P_1) og

$(2,0,0)(1,3,0)(3,5,0),(1,2,6)$ (P_2)

xy-planet:



V_1 = volumet til P_1

$$= \frac{1}{6}\Big|\big((2,0,0)-(0,0,0)\big)\times\big((1,3,0)-(0,0,0)\big)\cdot\big((1,2,6)-(0,0,0)\big)\Big|$$

$$= \frac{1}{6}\left|\begin{vmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 2 & 6 \end{vmatrix}\right| = \frac{1}{6}\underline{36} = \underline{\underline{6}}$$

V_2 = volumet til P_2

$$= \frac{1}{6}\Big|\big((1,3,0)-(2,0,0)\big)\times\big((3,5,0)-(2,0,0)\big)\cdot\big((1,2,6)-(2,0,0)\big)\Big|$$

$$= \frac{1}{6}\Big(\big((-1,3,0)\times(1,5,0)\big)\cdot(-1,2,6)\Big)$$

$$= \frac{1}{6}\left|\begin{vmatrix} -1 & 3 & 0 \\ 1 & 5 & 0 \\ -1 & 2 & 6 \end{vmatrix}\right| = \frac{1}{6}\left|-\begin{vmatrix} 5 & 0 \\ 2 & 6 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 \\ -1 & 6 \end{vmatrix} + 0\begin{vmatrix} 1 & 5 \\ -1 & 2 \end{vmatrix}\right|$$

$$= \frac{1}{6}\left|-30-18+0\right| = \frac{1}{6}\cdot 48 = 8$$

$\underline{\text{Volumet til } P = V_1 + V_2 = 14}$!

4a

$$\int_1^{\sqrt{2}} \frac{1+\tan^2 x}{\tan^2 x + 2\tan x + 2}\, dx = \int_{x=1}^{x=\sqrt{2}} \frac{du}{u^2+2u+2}$$

$u = \tan x$

$du = (1+\tan^2 x)\, dx$

$$= \int_{x=1}^{x=\sqrt{2}} \frac{du}{(u+1)^2+1}$$

$$= \Big[\arctan(u+1)\Big]_{x=1}^{x=\sqrt{2}}$$

$$= \Big[\arctan(\tan x + 1)\Big]_{x=1}^{\sqrt{2}}$$

$$= \arctan(\tan\sqrt{2} + 1) - \arctan(\tan 1 + 1).$$

b)

$$I = \int_1^{\infty} \frac{\sin x}{2x^{\frac{3}{2}}-1}\, dx$$

vil vise at I konvergerer ved sammenligningskriteriet.

$\frac{\sin x}{2x^{\frac{3}{2}}-1} \to 0$ når $x \to \infty$,

$\frac{\sin x}{2x^{\frac{3}{2}}-1}$ er kontinuerlig på $[1, \infty)$

og $-\frac{1}{x^{3/2}} < \frac{\sin x}{2x^{\frac{3}{2}}-1} < \frac{1}{x^{3/2}}$ når $x \in (1, \infty)$.

$\int_1^{\infty} \frac{1}{x^{3/2}}\, dx$ og $\int_1^{\infty} -\frac{1}{x^{3/2}}\, dx$ konvergerer, så da

konvergerer også $\int_1^{\infty} \frac{\sin x}{2x^{\frac{3}{2}}-1}\, dx$

5a $f: \mathbb{R} \to \mathbb{R}$ og $g: \mathbb{R} \to \mathbb{R}$ er henholdsvis kontinuerlig og deriverbar på $\mathbb{R}$

$$G(x) = \int_a^{g(x)} f(t)\,dt \qquad a \in \mathbb{R}$$

Setter $u = g(x)$, da er $G(u) = \int_a^u f(t)\,dt$.

Av fundamentalsetningen ~~og~~ kjerneregelen er

$$G'(x) = \underline{\frac{d}{dx} G(x)} = \frac{d}{du} G(u) \cdot \frac{d}{dx} u$$

$$= f(u) \cdot g'(x) = \underline{f(g(x)) \cdot g'(x)}$$

b) $$F_1(x) = \int_1^{e^x} (\ln t - 1)\,dt$$

av a) : $F_1'(x) = (\ln(e^x) - 1) \cdot (e^x)' = \underline{(x-1)e^x}$

$F_2 = \int_1^{x^2} (t^2 + t^4)\,dt \quad F_2'(x) = \frac{d}{dx}\left(\int_1^{x^2} (t^2 + t^4)\,dt\right) = (x^4 + x^8)(x^2)'$

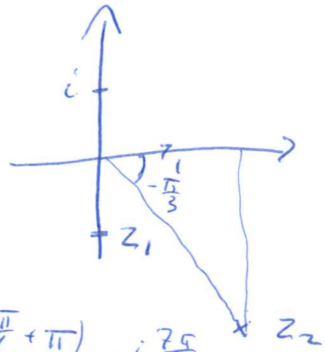
$$= \underline{2x^5 + 2x^9}$$

c) Både F_1 og F_2 er definert for alle $x \in \mathbb{R}$, med ~~og~~ kontinuerlige deriverte i hele $\mathbb{R}$.

~~Siden~~ En slik funksjon har en invers bare dersom den deriverte er ≥ 0 for alle x eller ≤ 0 for alle x.

Både F_1' og F_2' skifter fortegn, så ingen av dem har en invers.

6a $\qquad z_1 = -i \qquad\qquad z_2 = 2 - 2i\sqrt{3}$



på polarform

$$\underline{z_1 = e^{i\frac{3\pi}{2}}} \qquad \underline{z_2 = e^{i\frac{5\pi}{3}}}$$

$$w^2 = z_1 \Rightarrow w = \underline{e^{i\frac{3\pi}{4}}} \text{ eller } w = e^{i(\frac{3\pi}{4}+\pi)} = \underline{e^{i\frac{7\pi}{4}}}$$

$$\Rightarrow w = \underline{-\tfrac{1}{2}\sqrt{2} + \tfrac{i}{2}\sqrt{2}} \text{ eller } w = \underline{\tfrac{1}{2}\sqrt{2} - \tfrac{i}{2}\sqrt{2}}$$

$$w^2 = z_2 \Rightarrow w = e^{i\frac{5\pi}{6}} \text{ eller } w = e^{i(\frac{5\pi}{6}+\pi)} = e^{i\frac{11\pi}{6}}$$

$$\Rightarrow \underline{w = -\tfrac{1}{2}\sqrt{3} + i\tfrac{1}{2}} \text{ eller } \underline{w = \tfrac{1}{2}\sqrt{3} - \tfrac{i}{2}}$$

b) $\quad (z^2 - z_2)(z^2 - \bar{z}_2)$

$$= z^4 - (z_2 + \bar{z}_2)z^2 + z_2\bar{z}_2$$

$$= z^4 - (2 - 2i\sqrt{3} + 2 + 2i\sqrt{3})z^2 + (2 - 2i\sqrt{3})(2 + 2i\sqrt{3})$$

$$= z^4 - 4z^2 + (4 + 4\cdot 3) = \underline{z^4 - 4z^2 + 16}$$

er et fjerdegrads polynom som har røttene i $z^2 - z_2$ som røtter.

7.

$$f : [0,1] \to \mathbb{R}$$

$$f(x) = \begin{cases} \frac{1}{x} & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

er monotont avtagende på $(0,1)$

Siden f ikke er begrenset, er $\int_0^1 f(x)\,dx$ ikke definert.

(Det uegentlige integralet

$$\int_0^1 f(x)\,dx = \lim_{a \to 0} \int_a^1 f(x)\,dx$$ divergerer.)