Prøveeksamen Mat1100 høst 2020

Oppgave 1

$$\vec{F}'(x,y) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}$$

Så vi må ha

$$\left.\begin{aligned} \frac{\partial F_1}{\partial x} &= \sin x \quad \rightsquigarrow \quad -\cos x \\ \frac{\partial F_1}{\partial y} &= 1 \quad \rightsquigarrow \quad y \end{aligned}\right\} F_1(x,y) = y - \cos x$$

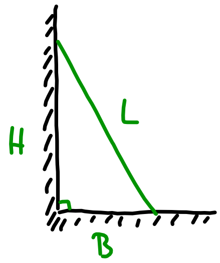
$$\left.\begin{aligned} \frac{\partial F_2}{\partial x} &= xy^2 \quad \rightsquigarrow \quad \tfrac{1}{2}x^2y^2 \\ \frac{\partial F_2}{\partial y} &= x^2y \quad \rightsquigarrow \quad \tfrac{1}{2}x^2y^2 \end{aligned}\right\} F_2(x,y) = \tfrac{1}{2}x^2y^2$$

Velger da $\vec{F}(x,y) = \underline{\underline{\left(y - \cos x,\ \tfrac{1}{2}x^2y^2\right)}}$

Oppgave 2

Velger
$$\begin{aligned} P(z) &= (z-2i)(z+2i)(z-(-3i))\cdot(z+(-3i)) \\ &= (z-2i)\cdot(z+2i)\cdot(z+3i)(z-3i) \\ &= (z^2+4)(z^2+9) \\ &= \underline{\underline{z^4 + 13z^2 + 36}} \end{aligned}$$

Oppgave 3



a) Pytagoras: $B^2 + H^2 = L^2$

b) Vi har $[B(t)]^2 + [H(t)]^2 = L^2$. Deriverer:

$$\cancel{2} \cdot B(t) \cdot B'(t) + \cancel{2} \cdot H(t) \cdot H'(t) = 0.$$

Vårt øyeblikk: $B'(t) = 2$ og $H(t) = 2 \cdot B(t)$. Innsatt:

$$\cancel{B(t)} \cdot 2 + 2 \cdot \cancel{B(t)} \cdot H'(t) = 0$$

$$2 + 2 \cdot H'(t) = 0$$

$$2 \cdot H'(t) = -2$$

$$\underline{\underline{H'(t) = -1 \quad \text{(meter per sekund)}}}$$

Oppgave 4

$$\int \frac{2\arcsin(2x) \cdot e^{\arcsin(2x)}}{\sqrt{1-4x^2}}\,dx$$

$u = \arcsin(2x)$ gir

$\frac{du}{dx} = \frac{1}{\sqrt{1-(2x)^2}} \cdot 2$

$du = \frac{2}{\sqrt{1-4x^2}}\,dx$

$F(u) = u \quad G'(u) = e^u$

$F'(u) = 1 \quad G(u) = e^u$

$$= \int \arcsin(2x)\, e^{\arcsin(2x)} \cdot \frac{2}{\sqrt{1-4x^2}}\,dx$$

$$= \int u e^u\,du$$

$$= ue^u - \int 1 \cdot e^u\,du = ue^u - e^u + C$$

$$= \underline{\underline{\arcsin 2x \cdot e^{\arcsin 2x} - e^{\arcsin 2x} + C}}$$

Integralet er alfså $\underline{\underline{\int \frac{2\arcsin(2x)\, e^{\arcsin(2x)}}{\sqrt{1-4x^2}}\,dx}}$

Oppgave 5

$$V = \int_1^2 2\pi x \cdot f(x)\,dx = 2\pi \int_1^2 x \cdot x^a\,dx$$

$$= 2\pi \int_1^2 x^{a+1}\,dx = 2\pi \left[\frac{1}{a+2} x^{a+2} \right]_{x=1}^{x=2}$$

$$= 2\pi \left[\frac{1}{a+2} \cdot 2^{a+2} - \frac{1}{a+2} \cdot \overset{1}{1^{a+2}} \right]$$

$$= \frac{2\pi}{a+2} \left[2^{a+2} - 1 \right]$$

$\boxed{a = 1}$

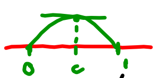
$$= \frac{2\pi}{3} \left[8 - 1 \right] = \frac{14\pi}{3}$$

Så $a = 1$

Oppgave 6

La $F(x) = \int_0^x e^{f(t)}\,dt$. Anta at $\int_0^1 e^{f(t)}\,dt = 0$.

Da er $F'(x) = e^{f(x)}$ for alle x, ved fundamentalteoremet. Vi har



$$F(0) = 0 \quad \text{og} \quad F(1) = 0.$$

Ved Rolles teorem fins da $c \in (0, 1)$ slik at $F'(c) = 0$

Men $F'(c) = e^{f(c)} > 0$ for alle c. Selvmotsigelse.

Altså fins ingen funksjon f som angitt i oppgaveteksten.

Oppgave 7

a)

$\text{leser} \xrightarrow{50\%} \text{leser}$, $\text{leser} \xrightarrow{50\%} \text{andre ting}$

$\text{sover} \xrightarrow{50\%} \text{leser}$, $\text{sover} \xrightarrow{50\%} \text{andre ting}$

$\text{andre ting} \xrightarrow{100\%} \text{sover}$

Alternativt:

$$\begin{cases} \text{leser} \longrightarrow \frac{1}{2}(\text{leser}) + \frac{1}{2}(\text{andre ting}) \\ \text{sover} \longrightarrow \frac{1}{2}(\text{leser}) + \frac{1}{2}(\text{andre ting}) \\ \text{andre ting} \longrightarrow 1\,(\text{sover}) \end{cases}$$

Dette gir

$$\begin{cases} x_{n+1} = (\text{antall som leser i morgen}) = \frac{1}{2}x_n + \frac{1}{2}y_n \\ y_{n+1} = (\text{--"-- sover --"--}) = z_n \\ z_{n+1} = (\text{--"-- gjør andre ting --"--}) = \frac{1}{2}x_n + \frac{1}{2}y_n \end{cases}$$

Så

$$x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}y_n + 0z_n$$
$$y_{n+1} = 0x_n + 0y_n + 1\cdot z_n$$
$$z_{n+1} = \frac{1}{2}x_n + \frac{1}{2}y_n + 0z_n$$

dvs.

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \cdot \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} \qquad M = \underline{\underline{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}}}$$

b)

$$M^2 = \begin{array}{ccc|ccc} & & & \frac{1}{2} & \frac{1}{2} & 0 \\ & & & 0 & 0 & 1 \\ & & & \frac{1}{2} & \frac{1}{2} & 0 \\ \hline \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{array} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$M^4 = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 5/16 & 5/16 & 3/8 \\ 3/8 & 3/8 & 1/4 \\ 5/16 & 5/16 & 3/8 \end{bmatrix}$$

Så: $N = \begin{bmatrix} \frac{5}{16} & \frac{5}{16} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \\ \frac{5}{16} & \frac{5}{16} & \frac{3}{8} \end{bmatrix}$

Dette gir $\begin{bmatrix} x_5 \\ y_5 \\ z_5 \end{bmatrix} = N \cdot \begin{bmatrix} 500 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2500}{16} \\ \frac{1500}{8} \\ \frac{2500}{16} \end{bmatrix} = \begin{bmatrix} 156{,}25 \\ 187{,}5 \\ 156{,}25 \end{bmatrix}$

Ifølge modellen er det ca. 156 som leser, 188 som sover og 156 som gjør andre ting på dag 5.

c) $\det M = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{vmatrix}$

$$= \frac{1}{2} \cdot \begin{vmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{vmatrix} - \frac{1}{2} \cdot \begin{vmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{vmatrix} + 0 \cdot \begin{vmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{2}\left(0 - \frac{1}{2}\right) - \frac{1}{2}\left(0 - \frac{1}{2}\right) + 0 = -\frac{1}{4} + \frac{1}{4} = 0$$

$\det N = \begin{vmatrix} \frac{5}{16} & \frac{5}{16} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \\ \frac{5}{16} & \frac{5}{16} & \frac{3}{8} \end{vmatrix}$

$$= \frac{5}{16} \begin{vmatrix} \frac{3}{8} & \frac{1}{4} \\ \frac{5}{16} & \frac{3}{8} \end{vmatrix} - \frac{5}{16} \begin{vmatrix} \frac{3}{8} & \frac{1}{4} \\ \frac{5}{16} & \frac{3}{8} \end{vmatrix} + \frac{3}{8} \begin{vmatrix} \frac{3}{8} & \frac{3}{8} \\ \frac{5}{16} & \frac{5}{16} \end{vmatrix}$$

$$= \frac{3}{8} \cdot (0) = 0$$

Så verken N eller M er inverterbare

Oppgave 8

$$f(x) = \begin{cases} \dfrac{\arctan x}{\sin x} & \text{for } x \neq 0 \\ K & \text{for } x = 0 \end{cases}$$

a) $$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\arctan x}{\sin x} \overset{\left[\frac{0}{0}\right]}{=} \lim_{x\to 0} \frac{\frac{1}{1+\overset{\to 0}{x^2}}}{\underset{\to 1}{\cos x}} = 1$$

Så hvis $K=1$, blir $\lim_{x\to 0} f(x) = f(0) = 1$. Altså er f kontinuerlig i $x=0$ når $K=1$

b) $$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)-1}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\arctan h}{\sin h} - 1}{h} = \lim_{h\to 0} \frac{\arctan h - \sin h}{h \sin h}$$

$$\overset{\left[\frac{0}{0}\right]}{=} \lim_{h\to 0} \frac{\frac{1}{1+h^2} - \cos h}{\sin h + h\cos h} = \lim_{h\to 0} \frac{1 - \cos h\,(1+h^2)}{(1+h^2)(\sin h + h\cos h)}$$

$$= \lim_{h\to 0} \frac{1 - \cos h - h^2\cos h}{\sin h + h\cos h + h^2 \sin h + h^3 \cos h}$$

$$\overset{\left[\frac{0}{0}\right]}{=} \lim_{h\to 0} \frac{\overset{\to 0}{\sin h - 2h\cos h + h^2 \sin h}}{\underset{\to 2}{\cos h + \cos h} \underset{\to 0}{- h\sin h + 2h\sin h + h^2\cos h + 3h^2\cos h - h^3 \sin h}}$$

$$= \frac{0}{2} = 0.$$

Så f er deriverbar i $x=0$, med $f'(0)=0$.