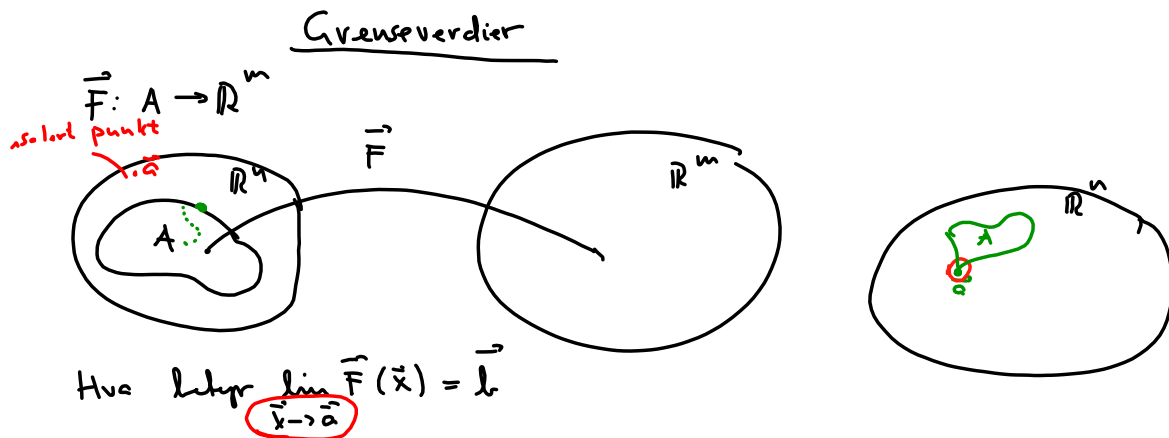




Definisjonen: Vi kaller et punkt $\vec{a}$ et opphopningspunkt for A dersom enhver kule $B(\vec{a}; \varepsilon)$ inneholder uendelig mange punkter fra A .

Definisjon: Anta at $A \subseteq \mathbb{R}^n$ og at $\vec{F}: A \rightarrow \mathbb{R}^m$. Dersom $\vec{a}$ er et opphopningspunkt for A , så ser vi at

$$\lim_{\vec{x} \rightarrow \vec{a}} \vec{F}(\vec{x}) = \vec{b}$$

dersom det for hver $\varepsilon > 0$ finnes en $\delta > 0$ slik at når $|\vec{x} - \vec{a}| < \delta$ og $\vec{x} \in A$, så er $|\vec{F}(\vec{x}) - \vec{b}| < \varepsilon$.

Regnesregler: Anta $\vec{F}, \vec{G}: A \rightarrow \mathbb{R}^m$ og at $\vec{a}$ er et opphopningspunkt for A . Dersom $\lim_{\vec{x} \rightarrow \vec{a}} \vec{F}(\vec{x}) = \vec{c}$ og $\lim_{\vec{x} \rightarrow \vec{a}} \vec{G}(\vec{x}) = \vec{d}$ finnes, så

$$(i) \lim_{\vec{x} \rightarrow \vec{a}} [\vec{F}(\vec{x}) + \vec{G}(\vec{x})] = \vec{c} + \vec{d}$$

$$(ii) \lim_{\vec{x} \rightarrow \vec{a}} [\vec{F}(\vec{x}) - \vec{G}(\vec{x})] = \vec{c} - \vec{d}$$

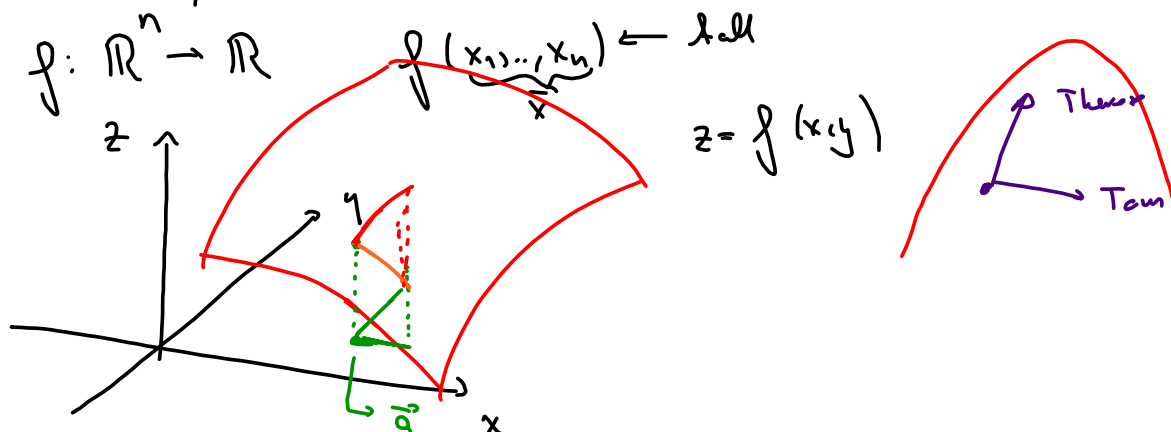
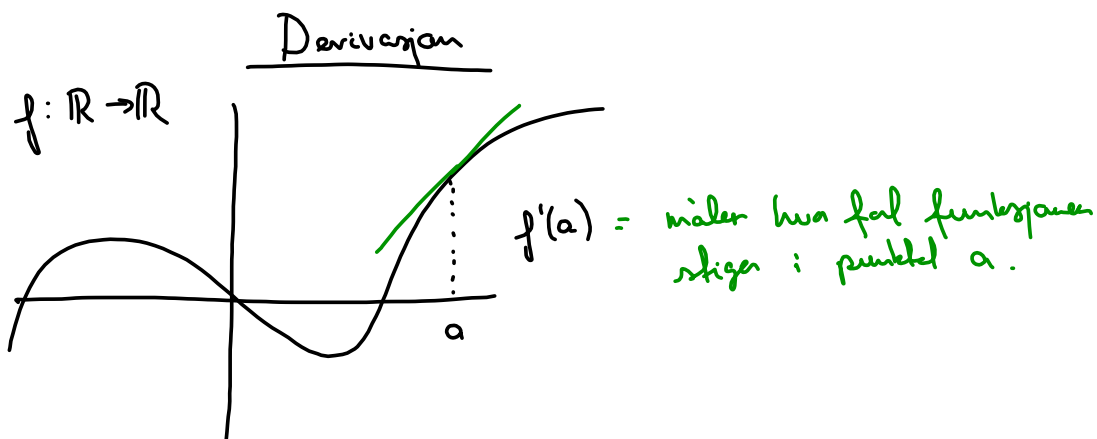
$$(iii) \lim_{\vec{x} \rightarrow \vec{a}} \vec{F}(\vec{x}) \cdot \vec{G}(\vec{x}) = \vec{c} \cdot \vec{d}$$

Husk $\vec{F}(\vec{x}) = \begin{pmatrix} F_1(\vec{x}) \\ F_2(\vec{x}) \\ \vdots \\ F_m(\vec{x}) \end{pmatrix}$

Satzung: Anta at $\vec{F}(\vec{x}) = \begin{pmatrix} F_1(\vec{x}) \\ F_2(\vec{x}) \\ \vdots \\ F_m(\vec{x}) \end{pmatrix}$ og at $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$

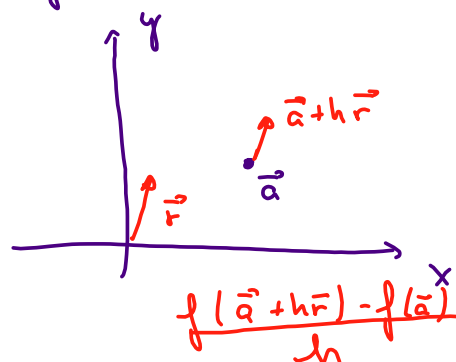
Da er $\lim_{\vec{x} \rightarrow \vec{a}} \vec{F}(\vec{x}) = \vec{c}$ det samme som at

$$\lim_{\vec{x} \rightarrow \vec{a}} F_1(\vec{x}) = c_1, \lim_{\vec{x} \rightarrow \vec{a}} F_2(\vec{x}) = c_2, \dots, \lim_{\vec{x} \rightarrow \vec{a}} F_m(\vec{x}) = c_m$$



Definisjon: Den retningsderiverte $f'(\vec{a}; \vec{r})$ i punktet $\vec{a}$ og retningen $\vec{r}$ er

$$f'(\vec{a}; \vec{r}) = \lim_{h \rightarrow 0} \frac{f(\vec{a} + h\vec{r}) - f(\vec{a})}{h}$$



Eksempel: $f(x, y) = x^2 y$

Finn $f'(\vec{a}; \vec{r})$ når $\vec{a} = (2, 1)$, $\vec{r} = (-1, 3)$.

Da er $\vec{a} + h\vec{r} = (2, 1) + h(-1, 3) = (2-h, 1+3h)$

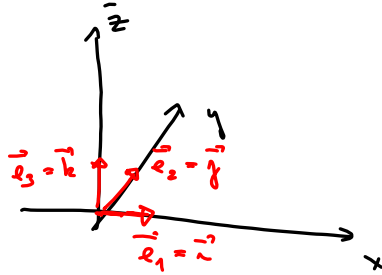
$$f'(\vec{a}; \vec{r}) = \lim_{h \rightarrow 0} \frac{(2-h)^2(1+3h) - 2^2 \cdot 1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(4 - 2 \cdot 2h + h^2)(1+3h) - 4}{h} = \lim_{h \rightarrow 0} \frac{(4 - 4h + h^2)(1+3h) - 4}{h}$$

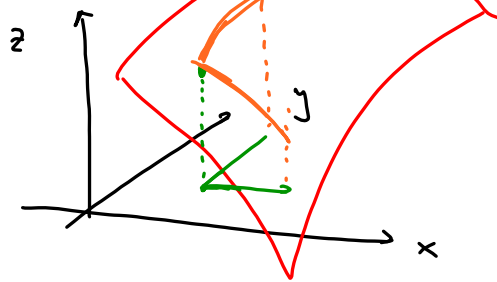
$$= \lim_{h \rightarrow 0} \frac{\cancel{4} + 12\cancel{h} - 4\cancel{h} - 12h^2 + h^2 + 3h^3 - 4}{h} = \underline{\underline{8}}$$

Partiellderivate

La $\vec{e}_1 = (1, 0, \dots, 0)$, $\vec{e}_2 = (0, 1, 0, \dots, 0)$, ..., $\vec{e}_i = (0, \dots, 1, \dots, 0)$
 $\uparrow$
i-te plass



Den partiellderivate $\frac{\partial f}{\partial x_i}(\vec{a}) = f'(\vec{a}, \vec{e}_i)$


 $\frac{\partial f}{\partial x}$
 $\frac{\partial f}{\partial y}$

Vi har:

$$\frac{\partial f}{\partial x_i}(\vec{a}) = f'(\vec{a}, \vec{e}_i) = \lim_{h \rightarrow 0} \frac{f(\vec{a} + h\vec{e}_i) - f(\vec{a})}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(a_1, a_2, \dots, a_i + h, \dots, a_n) - f(a_1, a_2, \dots, a_i, \dots, a_n)}{h}$$

*Den vanligste deriverte
m.h.p x_i mens alle
de andre variablene er
konstante.*

Eksempel: $f(x, y, z) = xye^{-z^2}$

$$\frac{\partial f}{\partial x} = ye^{-z^2} \quad (\text{deriverer m.h.p } x \text{ som om } y, z \text{ var konstante})$$

$$\frac{\partial f}{\partial y} = xe^{-z^2}$$

$$\frac{\partial f}{\partial z} = xy e^{-z^2} (-2z) = -2xyz e^{-z^2}$$

Definisjon: En funksjon $f(x_1, x_2, \dots, x_n)$ har n partiellderivate:

$$\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}. \quad \text{Gradienten til } f \text{ er vektorfunksjonen}$$

$$\nabla f(x_1, x_2) = \left(\frac{\partial f}{\partial x_1}(x_1, x_2), \frac{\partial f}{\partial x_2}(x_1, x_2), \dots, \frac{\partial f}{\partial x_n}(x_1, x_2, \dots, x_n) \right)$$

∇ eksempel ovenfor har:

$$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (ye^{-z^2}, xe^{-z^2}, -2xyz e^{-z^2})$$

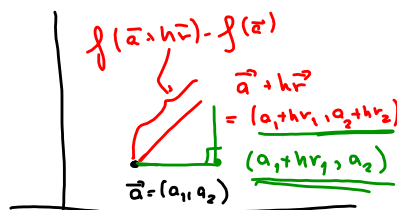
$$= e^{-z^2} (y, x, -2xyz)$$

Hvordan er sammenhengen mellom generelle retningsderiverte og partiellderiverte?

$$z = f(x_1, x_2)$$

$$f'(\vec{a}; \vec{r}) \text{ n\u00e5r } \vec{a} = (a_1, a_2), \vec{r} = (r_1, r_2)$$

$$f'(\vec{a}; \vec{r}) = \lim_{h \rightarrow 0} \frac{f(\vec{a} + h\vec{r}) - f(\vec{a})}{h}$$



Økning i funksjonsverdi

$$\underbrace{f(a_1 + hr_1, a_2) - f(a_1, a_2)}_{\text{Økning langs x-aksen}} + \underbrace{f(a_1 + hr_1, a_2 + hr_2) - f(a_1 + hr_1, a_2)}_{\text{Økning langs y-aksen}}$$

$$\approx \frac{\partial f}{\partial x}(a_1, a_2) hr_1 + \frac{\partial f}{\partial y}(a_1 + hr_1, a_2) hr_2$$

Dal ledder:

$$f'(\vec{a}; \vec{r}) \approx \lim_{h \rightarrow 0} \frac{\frac{\partial f}{\partial x}(a_1, a_2) hr_1 + \frac{\partial f}{\partial y}(a_1 + hr_1, a_2) hr_2}{h}$$

$$= \frac{\partial f}{\partial x}(a_1, a_2) r_1 + \frac{\partial f}{\partial y}(a_1, a_2) r_2$$

$$= \nabla f(a_1, a_2) \cdot \vec{r}$$

Føreløpssammenheng: For "smille" funksjoner er

$$f'(\vec{a}; \vec{r}) = \nabla f(\vec{a}) \cdot \vec{r}$$

Setning: Deriverte av partiellderiverte til f er kontinuerlige i punktet $\vec{a}$, så er $f'(\vec{a}; \vec{r}) = \nabla f(\vec{a}) \cdot \vec{r}$.

Eksempel: La $f(x, y) = x^2 e^{xy}$ og finn

$$f'(\vec{a}; \vec{r}) \text{ n\u00e5r } \vec{a} = (1, -2) \text{ og } \vec{r} = (1, 1)$$

Finne de partiellderiverte:

$$\frac{\partial f}{\partial x} = 2x e^{xy} + x^2 e^{xy} y = (2x + x^2 y) e^{xy} \quad \text{b\u00e5l.}$$

$$\frac{\partial f}{\partial y} = x^2 e^{xy} x = x^3 e^{xy} \quad \text{b\u00e5l.}$$

Siden $\frac{\partial f}{\partial x}$ og $\frac{\partial f}{\partial y}$ er kontinuerlige, s\u00e5 gjelder

$$f'(\vec{a}; \vec{r}) = \nabla f(\vec{a}) \cdot \vec{r}$$

$$\text{Siden } \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = ((2x + x^2 y) e^{xy}, x^3 e^{xy})$$

$$= e^{xy} (2x + x^2 y, x^3)$$

$$\nabla f(1, -2) = e^{1(-2)} (2 \cdot 1 + 1^2(-2), 1^3) = e^{-2} (0, 1)$$

$$f'(\vec{a}; \vec{r}) = \nabla f(1, -2) \cdot (1, 1) = e^{-2} (0, 1) \cdot (1, 1) = \underline{e^{-2}}$$