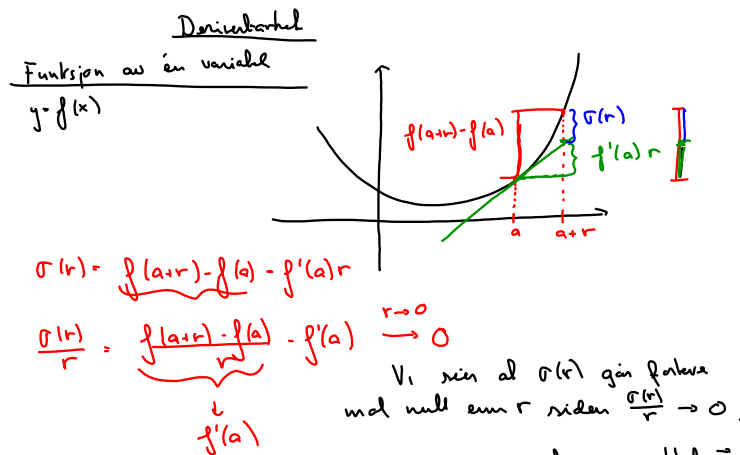




Definition: En funktion $f: \mathbb{R}^n \rightarrow \mathbb{R}$ kallas derivierbar i punkten $\vec{a}$ om det finns en linjär avbildning $df(\vec{a})$ sådan att

$$\sigma(\vec{r}) = f(\vec{a} + \vec{r}) - f(\vec{a}) - df(\vec{a}) \cdot \vec{r}$$

går mot 0 snabbare än $\|\vec{r}\|$, dvs $\lim_{\vec{r} \rightarrow 0} \frac{\sigma(\vec{r})}{\|\vec{r}\|} = 0$.

Sats: Derasom f är derivierbar i $\vec{a}$, så

$$f'(\vec{a}; \vec{r}) = df(\vec{a}) \cdot \vec{r}$$

Beweis:

$$\lim_{h \rightarrow 0} \frac{f(\vec{a} + h\vec{r}) - f(\vec{a})}{h} = \lim_{h \rightarrow 0} \frac{df(\vec{a}) \cdot (h\vec{r}) + \sigma(h\vec{r})}{h}$$

$$= \lim_{h \rightarrow 0} \left(df(\vec{a}) \cdot \vec{r} + \frac{\sigma(h\vec{r})}{h} \right) = df(\vec{a}) \cdot \vec{r}$$

$\downarrow 0$

Sats: Derasom f är derivierbar, så är f kontinuerlig i $\vec{a}$.

Beweis:

$$\lim_{\vec{r} \rightarrow 0} f(\vec{a} + \vec{r}) = \lim_{\vec{r} \rightarrow 0} [f(\vec{a}) + df(\vec{a}) \cdot \vec{r} + \sigma(\vec{r})] = f(\vec{a})$$

$\leq |df(\vec{a})| \|\vec{r}\| \rightarrow 0$

Teorem: Derasom alla de partiella derivata till f är definierade i en omgivning om $\vec{a}$ och kontinuerliga i $\vec{a}$, så är f derivierbar i $\vec{a}$.

Minner om mitterledsberäkningar:

Beweis (för funktioner av 2 variabler)

$$\sigma(\vec{r}) = f(\vec{a} + \vec{r}) - f(\vec{a}) - df(\vec{a}) \cdot \vec{r}$$

Vi har $f(\vec{a} + \vec{r}) = f(a_1 + r_1, a_2 + r_2)$

$$= f(a_1 + r_1, a_2 + r_2) - f(a_1, a_2)$$

$$= \underbrace{f(a_1 + r_1, a_2 + r_2) - f(a_1 + r_1, a_2)}_{\text{I}} + \underbrace{f(a_1 + r_1, a_2) - f(a_1, a_2)}_{\text{II}}$$

$$= \left\{ \frac{\partial f}{\partial y}(a_1, r_1) r_2 + \frac{\partial f}{\partial x}(a_1, a_2) r_1 \right\}$$

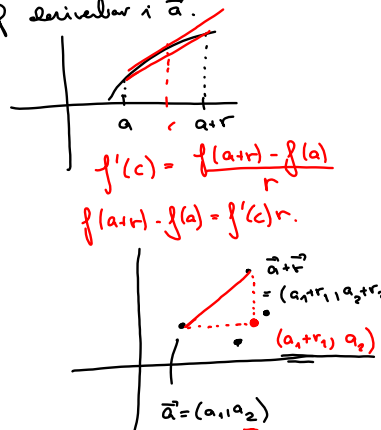
Derasom $\vec{r} = (r_1, r_2)$

$$\sigma(\vec{r}) = \frac{\partial f}{\partial x}(a_1, a_2) r_1 + \frac{\partial f}{\partial y}(a_1, r_1) r_2 - \left(\frac{\partial f}{\partial x}(a_1, a_2), \frac{\partial f}{\partial y}(a_1, a_2) \right) \cdot (r_1, r_2)$$

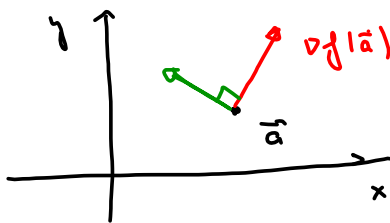
$$= \frac{\partial f}{\partial x}(a_1, a_2) r_1 + \frac{\partial f}{\partial y}(a_1, r_1) r_2 - \frac{\partial f}{\partial x}(a_1, a_2) r_1 - \frac{\partial f}{\partial y}(a_1, a_2) r_2$$

$$= \left[\frac{\partial f}{\partial x}(a_1, r_1) - \frac{\partial f}{\partial x}(a_1, a_2) \right] r_1 + \left[\frac{\partial f}{\partial y}(a_1, r_1) - \frac{\partial f}{\partial y}(a_1, a_2) \right] r_2$$

Derasom $\frac{\sigma(\vec{r})}{\|\vec{r}\|} = \frac{(\rightarrow 0) r_1 + (0 \rightarrow) r_2}{\|\vec{r}\|} = 0$



Sætning: Hvis f er differentierbar i $\vec{a}$, så peger $\nabla f(\vec{a})$ i den retningen hvor funktionen øger størst og størrelsen i den retningen er $|\nabla f(\vec{a})|$



Beweis: La $\vec{u}$ være en enhedsvektor:

$$f'(\vec{a}; \vec{u}) = \nabla f(\vec{a}) \cdot \vec{u} \leq |\nabla f(\vec{a})| \cdot |\vec{u}| = |\nabla f(\vec{a})|$$

Schwach
= Længde af $\nabla f(\vec{a})$ i $\vec{u}$

Eksamen 2021: $f(x, y) = x^2 y$

a) I hvilken retning vokser f vækst i punktet $(-4, 1)$?

$$\frac{\partial f}{\partial x} = 2xy \quad \frac{\partial f}{\partial y} = x^2 \quad \nabla f(x, y) = (2xy, x^2)$$

$$\nabla f(-4, 1) = (2 \cdot (-4) \cdot 1, (-4)^2) = (-8, 16) = 8(-1, 2)$$

Størrelse: $|\nabla f(-4, 1)| = |8(-1, 2)| = 8|(-1, 2)| = 8\sqrt{(-1)^2 + 2^2} = 8\sqrt{5}$

b) Find $f'(\vec{a}, \vec{r})$ når $\vec{a} = (-4, 1)$, $\vec{r} = (2, 3)$

Siden de partiellderivater er konstante, er f differentierbar og

$$f'(\vec{a}, \vec{r}) = \nabla f(\vec{a}) \cdot \vec{r} = (-8, 16) \cdot (2, 3) = 8(-1, 2) \cdot (2, 3) = 8[-2 + 6] = 32$$

Find en vektor $\vec{s} \neq 0$ så $f'(\vec{a}, \vec{s}) = 0$

Prøv $\vec{s} = (2, 1)$ (når $\vec{s} \perp \nabla f(\vec{a})$):

$$f'(\vec{a}, \vec{s}) = \nabla f(\vec{a}) \cdot \vec{s} = (-8, 16) \cdot (2, 1) = -16 + 16 = 0$$

Höjere ordens deriverte

$$f(x, y) \Rightarrow \frac{\partial f}{\partial x} \text{ og } \frac{\partial f}{\partial y}$$

Andre deriverte: $\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2}$

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2}$$

Eksempel: $f(x, y) = xy + x$, $\frac{\partial f}{\partial x} = y + 1$, $\frac{\partial f}{\partial y} = x$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (y + 1) = \underline{0}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (y + 1) = \underline{1}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} (x) = \underline{1}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} (x) = \underline{0}$$

Sætning: Hvis $f(x_1, \dots, x_n)$ er en derivert funksjon av n variable og $x_i \neq x_j$ er to forskjellige variable, så er

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

forutsatt at disse storskene er kontinuerlige.

Hva med høyere ordens partiell deriverte?

$$f(x, y, z, w)$$

$$\frac{\partial^4 f}{\partial z \partial x \partial z \partial w}$$

$$\frac{\partial^4 f}{\partial w \partial z \partial x \partial z}$$

(derivert 4 ganger, først m.h.p w , så m.h.p z , så m.h.p x , og så m.h.p z)

Disse partiell deriverte er like hvis de er kontinuerlige.

Derivasjon av vektorvaluede funksjoner

$$\vec{F} : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\vec{F}(\vec{x}) = \begin{pmatrix} F_1(\vec{x}) \\ F_2(\vec{x}) \\ \vdots \\ F_m(\vec{x}) \end{pmatrix} \quad F_1 : \mathbb{R}^n \rightarrow \mathbb{R}$$

$$\nabla f(\vec{a}) = \left(\frac{\partial f}{\partial x_1}(\vec{a}), \dots, \frac{\partial f}{\partial x_n}(\vec{a}) \right)$$

Den "deriverte til $\vec{F}$ " er Jacobi-matrisen:

$$\vec{F}'(\vec{a}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(\vec{a}) & \frac{\partial F_1}{\partial x_2}(\vec{a}) & \dots & \frac{\partial F_1}{\partial x_n}(\vec{a}) \\ \frac{\partial F_2}{\partial x_1}(\vec{a}) & \frac{\partial F_2}{\partial x_2}(\vec{a}) & \dots & \frac{\partial F_2}{\partial x_n}(\vec{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1}(\vec{a}) & \frac{\partial F_m}{\partial x_2}(\vec{a}) & \dots & \frac{\partial F_m}{\partial x_n}(\vec{a}) \end{pmatrix} = \begin{pmatrix} \nabla F_1(\vec{a}) \\ \nabla F_2(\vec{a}) \\ \vdots \\ \nabla F_m(\vec{a}) \end{pmatrix}$$

Eksempel: $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \quad \vec{F}(x, y, z) = \begin{pmatrix} xy+z \\ x-z \end{pmatrix}$

$$\vec{F}'(x, y, z) = \begin{pmatrix} \frac{\partial}{\partial x}(xy+z) & \frac{\partial}{\partial y}(xy+z) & \frac{\partial}{\partial z}(xy+z) \\ \frac{\partial}{\partial x}(x-z) & \frac{\partial}{\partial y}(x-z) & \frac{\partial}{\partial z}(x-z) \end{pmatrix} = \begin{pmatrix} y & x & 1 \\ 1 & 0 & -1 \end{pmatrix}$$