Chain rule

$$(f\circ g)'(x) = f'(g(x))\,g'(x)$$

$$g(x+h) = g(x) + g'(x)h + \psi(h)$$
$$f(y+k) = f(y) + f'(y)k + \phi(k) \qquad y = g(x)$$

$$\begin{aligned} f(g(x+h)) &= f\big(g(x) + g'(x)h + \psi(h)\big) \\ &= f(g(x)) + f'(g(x))\big(g'(x)h + \psi(h)\big) \\ &\quad + \phi\big(g'(x)h + \psi(h)\big) \\ &= f(g(x)) + f'(g(x))\,g'(x)h \\ &\quad + f'(g(x))\,\psi(h) + \phi\big(g'(x)h + \psi(h)\big) \end{aligned}$$

prove: $\lim_{h\to 0} \phi\big(g'(x)h + \psi(h)\big)/h = 0$, according to ε-δ definitions.

We know $\lim_{t\to 0} \dfrac{\phi(t)}{t} = 0$

also $|g'(x)h + \psi(h)| \le Ch$, for h small enough, some C.
eg $C = |g'(x)| + 1$.

Let $\varepsilon > 0$. Choose $\delta > 0$ s.t.:

$$\left|\frac{\phi(t)}{t}\right| < \frac{\varepsilon}{C} \quad \text{for } |t| < \delta$$

then, for $h < \dfrac{\delta}{C}$ we have $|g'(x)h + \psi(h)| \le Ch < \delta$

so $|\phi(g'(h) + \psi(h))| \le \dfrac{\varepsilon}{C}|g'(x)h + \psi(h)|$

$\le \dfrac{\varepsilon}{C} Ch \le \varepsilon h$

derivative of inverse

f continuous and monotone

$f'(x) \neq 0$

then f^{-1} differentiable at $f(x)$ and $\left(f^{-1}\right)'(f(x)) = \dfrac{1}{f'(x)}$

necessary from chain rule (value) : $f^{-1} \circ f = id$: $\left(f^{-1}\right)'(f(x)) f'(x) = 1$

$$\frac{f^{-1}\big(f(x)+h\big) - \overbrace{f^{-1}(f(x))}^{x}}{h} \xrightarrow{\;?\;} \frac{1}{f'(x)}$$

$k = f^{-1}(f(x)+h) - x$ means $f(x+k) = f(x)+h$

so $h = f(x+k) - f(x)$

$k(h)$: is a function, defined for small h. It's continuous at 0.

$$\lim_{h\to 0} \frac{k(h)}{h} = \lim_{h\to 0} \frac{1}{\dfrac{f(x+k(h)) - f(x)}{k(h)}}$$

facts 1) : $k(h) \xrightarrow{h\to 0} 0$ (by continuity of f^{-1} (separate theorem))

2) $\phi(\ell) = \begin{cases} f'(x) & \ell = 0 \\ \dfrac{f(x+\ell) - f(x)}{\ell} & \ell \neq 0 \end{cases}$ then ϕ is continuous at 0.

$\rightsquigarrow$ composition of continuous functions : $\phi \circ k$ is continuous

$$\lim_{h\to 0} \phi \circ k(h) = \phi(k(0)) = f(x)$$