LØSNINGSFORSLAG ; EKSAMEN I

MAT1100U 24. april 2013

DEL 2 :

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11) $\lim\limits_{x\to 0} \frac{1}{x}\int_0^x (1+\sin t)^{1/t}\,dt$

$$= \lim_{x\to 0} \frac{\int_0^x (1+\sin t)^{1/t}\,dt}{x} \overset{\frac{0}{0}\ \text{L'H}}{\underset{\text{AFT}}{=}} \lim_{x\to 0} \frac{(1+\sin x)^{1/x}}{1}$$

$$\boxed{\begin{aligned} \lim_{x\to 0} \ln(1+\sin x)^{1/x} &= \lim_{x\to 0} \frac{1}{x}\ln(1+\sin x) \\ &= \lim_{x\to 0} \frac{\ln(1+\sin x)}{x} \overset{\frac{0}{0}}{\underset{\text{L'H}}{=}} \lim_{x\to 0} \frac{\frac{1}{1+\sin x}\cdot\cos x}{1} \\ &= \lim_{x\to 0} \frac{\cos x}{1+\sin x} = 1 \quad (*) \end{aligned}}$$

$$= \lim_{x\to 0} \left(e^{\ln(1+\sin x)}\right)^{1/x} \underset{\substack{\uparrow \\ e^x \text{ kontinuerlig}}}{=} e^{\lim_{x\to 0} \ln(1+\sin x)^{1/x}} \underset{\substack{\uparrow \\ (*)}}{=} e^1 = \underline{\underline{e}}$$

12) a) $\dfrac{1}{x^3-1} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1}$

Ser at $x=1$ er en rot; $x^3-1 : x-1 = x^2+x+1$

$1 = Ax^2 + Ax + A + Bx^2 - Bx + Cx - C$

$$\begin{cases} A+B=0 \\ A-B+C=0 \\ A-C=1 \end{cases}\Bigg\} \quad \begin{array}{r} 2A+C=0 \\ A-C=1 \\ \hline 3A=1 \end{array}$$

$\underline{A=\frac{1}{3}} \quad \underline{B=-\frac{1}{3}} \quad \underline{C=-\frac{2}{3}}$

$$\dfrac{1}{x^3-1} = \dfrac{\frac{1}{3}}{x-1} + \dfrac{-\frac{1}{3}x-\frac{2}{3}}{x^2+x+1}$$

b) $\displaystyle\int \frac{1}{x^3-1}\,dx = \frac{1}{3}\int\frac{1}{x-1}\,dx - \frac{1}{3}\int\frac{x+2}{x^2+x+1}\,dx$ $\boxed{\text{v/a)}}$

$\displaystyle = \frac{1}{3}\ln|x-1| - \frac{1}{6}\int\frac{2x+4}{x^2+x+1}\,dx$

$\displaystyle\int\frac{2x+4}{x^2+x+1}\,dx = \int\frac{2x+1}{x^2+x+1}\,dx + \int\frac{3}{x^2+x+1}\,dx$

$\displaystyle = \ln(x^2+x+1) + 3\int\frac{1}{x^2+x+1}\,dx$

$\displaystyle \overset{(*)}{=} \ln(x^2+x+1) + 3\int\frac{1}{\frac{3}{4}\left(\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right)}\,dx$

$\displaystyle = \ln(x^2+x+1) + 4\int\frac{1}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}\,dx$

$\displaystyle = \ln(x^2+x+1) + 2\sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$

(*) x^2+x+1
$= x^2+x+\frac{1}{4}+\frac{3}{4}$
$= \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$
$= \frac{3}{4}\left(\frac{\left(x+\frac{1}{2}\right)^2}{\frac{3}{4}}+1\right)$
$= \frac{3}{4}\left(\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right)$

$u = \frac{2x+1}{\sqrt{3}}$

$\frac{\sqrt{3}}{2}\,du = dx$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) - \frac{\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

13) $\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = M \begin{bmatrix} x_n \\ y_n \end{bmatrix}$, $n \geq 0$ der $M = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$

$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ $\quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 16 \\ 2 \end{bmatrix}$ $\quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 19 \\ 8 \end{bmatrix}$

a) $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = M \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

$\begin{bmatrix} 16 \\ 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

gir $\begin{cases} 16 = 4a + 8b \\ 2 = 4c \end{cases}$

$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = M \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

$\begin{bmatrix} 19 \\ 8 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \begin{bmatrix} 16 \\ 2 \end{bmatrix}$

gir

$\begin{cases} 19 = 16a + 2b \\ 8 = 2c \end{cases}$

Vi får $\underline{c = \frac{1}{2}}$ og $\begin{cases} 4a + 8b = 16 & | \cdot -4 \\ 16a + 2b = 19 \end{cases}$

$$\begin{cases} -16a - 32b = -64 \\ 16a + 2b = 19 \end{cases}$$

$$-30b = -45$$

$$\underline{b = \frac{3}{2}} \qquad a = \frac{16 - 8 \cdot \frac{3}{2}}{4} = \underline{1}$$

$$\underline{\underline{M = \begin{bmatrix} 1 & \frac{3}{2} \\ \frac{1}{2} & 0 \end{bmatrix}}}$$

b) Vis at $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} -5(-\frac{1}{2})^n + 9(\frac{3}{2})^n \\ 5(-\frac{1}{2})^n + 3(\frac{3}{2})^n \end{bmatrix}$ for $n \geq 0$ (*)

Bruker induksjon:

$n=0$: $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -5+9 \\ 5+3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ ok

Anta ok for n.

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} -5(-\frac{1}{2})^n + 9(\frac{3}{2})^n \\ 5(-\frac{1}{2})^n + 3(\frac{3}{2})^n \end{bmatrix}$$

siden $\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = M \begin{bmatrix} x_n \\ y_n \end{bmatrix}$
M er funnet i a)
og $\begin{bmatrix} x_n \\ y_n \end{bmatrix}$ er gitt av
induksjonshypotesen

$$= \begin{bmatrix} -5(-\frac{1}{2})^n + 9(\frac{3}{2})^n + 5 \cdot \frac{3}{2}(-\frac{1}{2})^n + 3(\frac{3}{2})(\frac{3}{2})^n \\ -5(\frac{1}{2})(-\frac{1}{2})^n + 9(\frac{1}{2})(\frac{3}{2})^n \end{bmatrix}$$

$$= \begin{bmatrix} 10(-\frac{1}{2})^{n+1} - 15(-\frac{1}{2})^{n+1} + 6(\frac{3}{2})^{n+1} + 3(\frac{3}{2})^{n+1} \\ 5(-\frac{1}{2})^{n+1} + 3(\frac{3}{2})^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} -5(-\frac{1}{2})^{n+1} + 9(\frac{3}{2})^{n+1} \\ 5(-\frac{1}{2})^{n+1} + 3(\frac{3}{2})^{n+1} \end{bmatrix},$$

så (*) er vist ved induksjon.

c) $$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{-5(-\frac{1}{2})^n + 9(\frac{3}{2})^n}{5(-\frac{1}{2})^n + 3(\frac{3}{2})^n}$$

$$= \lim_{n\to\infty} \frac{-5\left(\frac{-\frac{1}{2}}{\frac{3}{2}}\right)^n + 9}{5\left(\frac{-\frac{1}{2}}{\frac{3}{2}}\right)^n + 3} = \lim_{n\to\infty} \frac{-5(\frac{1}{3})^n + 9}{5(-\frac{1}{3})^n + 3} = 3$$

14) La $f:[0,2]\to\mathbb{R}$ være en kontinuerlig funksjon. Anta at f er to ganger deriverbar på intervallet $(0,2)$ og anta at $f(0)=0$, $f(1)=1$ og $f(2)=2$.

Da fins $c\in(0,2)$ slik at $f''(c)=0$.

Bevis: Siden f er kontinuerlig på $[0,1]$ og deriverbar på $(0,1)$, fins $\alpha\in(0,1)$ slik at

$$\frac{f(1)-f(0)}{1-0}=f'(\alpha) \text{ ved Middelverdisetningen,}$$

dvs. $f'(\alpha)=1$.

Siden f er kont. på $[1,2]$ og deriverbar på $(1,2)$, fins $\beta\in(1,2)$ slik at

$$\frac{f(2)-f(1)}{2-1}=f'(\beta) \text{ ved Middelverdisetningen,}$$

dvs. $f'(\beta)=1$.

Siden f' er kont. på $[\alpha,\beta]$ og deriverbar på (α,β), og $f'(\alpha)=f'(\beta)$ fins $c\in(\alpha,\beta)$ slik at

$$f''(c)=0 \text{ ved Rolles teorem.}$$

Siden $(\alpha,\beta)\subset(0,2)$, fins $c\in(0,2)$ slik at $f''(c)=0$. ∎

15) La $A \subset \mathbb{R}^n$. Anta at $\vec{F}: A \to \mathbb{R}^2$ er en funksjon i n variable gitt ved

$$\vec{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x})).$$

La $\vec{a} \in A$ og anta at F_1 og F_2 er kontinuerlige i $\vec{a}$. Vi vil vise at $\vec{F}$ er kont. i $\vec{a}$, dvs. vi vil vise at for hver $\varepsilon > 0$ fins $\delta > 0$ slik at

$$|\vec{F}(\vec{x}) - \vec{F}(\vec{a})| < \varepsilon \quad \text{for alle } \vec{x} \in A \text{ slik at } |\vec{x} - \vec{a}| < \delta.$$

$$\vec{F}(\vec{x}) - \vec{F}(\vec{a}) = (F_1(\vec{x}) - F_1(\vec{a}), F_2(\vec{x}) - F_2(\vec{a}))$$

$$|\vec{F}(\vec{x}) - \vec{F}(\vec{a})| = \sqrt{(F_1(\vec{x}) - F_1(\vec{a}))^2 + (F_2(\vec{x}) - F_2(\vec{a}))^2}$$

Gitt $\varepsilon > 0$.

Siden F_1 er kont. i $\vec{a}$ fins δ_1 slik at $|F_1(\vec{x}) - F_1(\vec{a})| < \frac{\varepsilon}{\sqrt{2}}$ for alle $\vec{x} \in A$ slik at $|\vec{x} - \vec{a}| < \delta_1$.

Siden F_2 er kont. i $\vec{a}$ fins δ_2 slik at $|F_2(\vec{x}) - F_2(\vec{a})| < \frac{\varepsilon}{\sqrt{2}}$ for alle $\vec{x} \in A$ slik at $|\vec{x} - \vec{a}| < \delta_2$.

La $\delta = \min(\delta_1, \delta_2)$.

Da er

$$|\vec{F}(\vec{x}) - \vec{F}(\vec{a})| = \sqrt{(F_1(\vec{x}) - F_1(\vec{a}))^2 + (F_2(\vec{x}) - F_2(\vec{a}))^2}$$

$$< \sqrt{\left(\frac{\varepsilon}{\sqrt{2}}\right)^2 + \left(\frac{\varepsilon}{\sqrt{2}}\right)^2} = \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \varepsilon$$

for alle $\vec{x} \in A$ slik at $|\vec{x} - \vec{a}| < \delta$.

Dermed er $\vec{F}$ kontinuerlig i $\vec{a}$. ∎