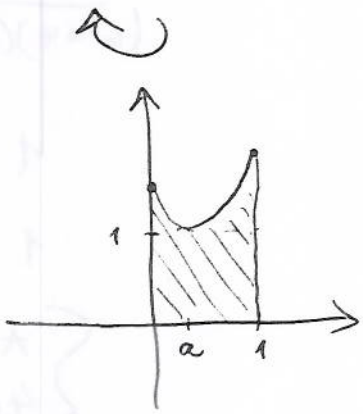
Løsningsforslag, del 2

eksamen MAT1100U 29/4-14

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11) $f:[0,1] \to \mathbb{R}$

$f(x) = (x-a)^2 + 1 \quad , \quad 0 \le a \le 1$



Om y-aksen:

$$V(a) = \int_0^1 2\pi x \left((x-a)^2 + 1\right) dx$$

$$= 2\pi \int_0^1 \left(x^3 - 2ax^2 + (a^2+1)x\right) dx$$

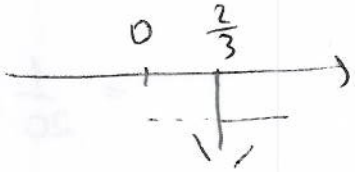
$$= 2\pi \left[\frac{1}{4}x^4 - \frac{2a}{3}x^3 + \frac{1}{2}(a^2+1)x^2\right]_0^1$$

$$= 2\pi \left(\frac{1}{4} - \frac{2a}{3} + \frac{1}{2}(a^2+1)\right)$$

$V'(a) = 0$ gir

$$-\frac{2}{3} + a = 0$$

$$a = \frac{2}{3}$$



gir minst volum

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12) a) Finn en delbrøkoppspalting av

$$\frac{1}{(u^2+4)(u+4)}$$

$$\frac{1}{(u^2+4)(u+4)} = \frac{Au+B}{u^2+4} + \frac{C}{u+4}$$

$$1 = (Au+B)(u+4) + C(u^2+4)$$

$$1 = Au^2 + Bu + 4Au + 4B + Cu^2 + 4C$$

$$\begin{cases} A+C=0 \\ 4A+B=0 \\ 4B+4C=1 \end{cases}$$

$A = -C$

$$\begin{cases} -4C+B=0 \\ 4B+4C=1 \end{cases}$$

$5B = 1$

$B = \frac{1}{5}$

$4A = -\frac{1}{5}$

$A = -\frac{1}{20}$

$C = \frac{1}{20}$

$$\frac{1}{(u^2+4)(u+4)} = \frac{-\frac{1}{20}u + \frac{1}{5}}{u^2+4} + \frac{\frac{1}{20}}{u+4}$$

$$= \frac{1}{20}\left(\frac{1}{u+4} - \frac{u-4}{u^2+4}\right)$$

b) Finn integralet

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{(\sin^2 x+4)(\sin x+4)}\,dx$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{(\sin^2 x+4)(\sin x+4)}\,dx$$

$u = \sin x$

$du = \cos x\,dx$

$x=0 \quad u=0$

$x=\frac{\pi}{2} \quad u=1$

$$= \int_0^1 \frac{1}{(u^2+4)(u+4)}\,du$$

vedl a)

$$= \frac{1}{20}\int_0^1 \left(\frac{1}{u+4} - \frac{u-4}{u^2+4}\right)du$$

$$= \frac{1}{20}\int_0^1 \frac{1}{u+4}\,du - \frac{1}{40}\int_0^1 \frac{2u}{u^2+4}\,du + \frac{1}{5}\int_0^1 \frac{1}{u^2+4}\,du$$

$$= \left[\frac{1}{20}\ln|u+4| - \frac{1}{40}\ln(u^2+4) + \frac{1}{10}\arctan\left(\frac{u}{2}\right)\right]_0^1$$

$$\int \frac{1}{u^2+4}\,du = \int \frac{1}{4}\left(\frac{1}{\left(\frac{u}{2}\right)^2+1}\right)du$$

$$= \frac{1}{2}\arctan\left(\frac{u}{2}\right)+C$$

$$= \frac{1}{20}\ln 5 - \frac{1}{40}\ln 5 + \frac{1}{10}\arctan\left(\frac{1}{2}\right) - \frac{1}{20}\ln 4 + \frac{1}{40}\ln 4$$

$$= \frac{1}{40}\ln 5 + \frac{1}{10}\arctan\left(\frac{1}{2}\right) - \frac{1}{40}\ln 4$$

$$= \frac{1}{10}\arctan\left(\frac{1}{2}\right) + \frac{1}{40}\ln\frac{5}{4}$$

13) $f(a+ib) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ $\quad a,b \in \mathbb{R}$

a) Vis at $f(zw) = f(z)f(w)$ for alle $z, w \in \mathbb{C}$

La $z = a+ib$, $w = c+id$

$$f((a+ib)(c+id)) = f((ac-bd)+i(bc+ad))$$

$$= \begin{bmatrix} ac-bd & bc+ad \\ -(bc+ad) & ac-bd \end{bmatrix} \quad (*)$$

$$f(a+ib)f(c+id) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} c & d \\ -d & c \end{bmatrix}$$

$$= \begin{bmatrix} ac-bd & ad+bc \\ -bc-ad & -bd+ac \end{bmatrix} \quad (**)$$

Vi ser at $(*) = (**)$. ∎

b) Vis at $f(z^n) = (f(z))^n$ for alle $z \in \mathbb{C}$, $n = 1,2,\ldots$

$n=1$: $f(z) = f(z)$ ok

Anta $f(z^{n-1}) = (f(z))^{n-1}$ $(\circ)$

$f(z^n) = f(z \cdot z^{n-1})$

$\overset{\text{ved a)}}{=} f(z) \cdot f(z^{n-1}) \overset{\text{ved}(\circ)}{=} f(z)\cdot(f(z))^{n-1} = (f(z))^n$

så påstanden er vist ved induksjon ∎

c) $\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}^9 = (f(\sqrt{3}+i))^9 = (f(2e^{i\frac{\pi}{6}}))^9$

$\sqrt{3}+i = 2(\frac{\sqrt{3}}{2}+\frac{i}{2}) = 2e^{i\frac{\pi}{6}}$

$\overset{\text{ved b)}}{=} f((2e^{i\frac{\pi}{6}})^9) = f(2^9 e^{i\frac{3\pi}{2}})$

$= f(-2^9 i) = \begin{bmatrix} 0 & -2^9 \\ 2^9 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -512 \\ 512 & 0 \end{bmatrix}$

14) $f:[a,b] \to \mathbb{R}$ kont. , $0<a<b$

der $\int_a^b f(x)\,dx = 0$

Vis at det fins $c \in (a,b)$ slik at $\int_a^c f(x)\,dx = cf(c)$.

La $F(x) = \frac{1}{x}\int_a^x f(t)\,dt$

F er kont. på $[a,b]$ (produkt av kont. funksjoner; $\int_a^x f(t)\,dt$ er deriverbar, så kont.) og deriverbar på (a,b).

Vi har $F(a)=0$, og $F(b)=0$ per antagelse

Ved middelverdisetningen fins c slik at

$$\frac{F(b)-F(a)}{b-a} = F'(c)$$

$$0 = F'(c)$$

$$F'(x) = -\frac{1}{x^2}\int_a^x f(t)\,dt + \frac{1}{x}f(x)$$

$$F'(c) = -\frac{1}{c^2}\int_a^c f(t)\,dt + \frac{1}{c}f(c)$$

$$\int_a^c f(t)\,dt = cf(c)$$

$$\int_a^c f(x)\,dx = cf(c) \quad \blacksquare$$

15) Vi har at $\lim_{x\to\pm\infty} |P(x)| = \infty$. Videre er

$$\lim_{x\to-\infty} \frac{|P(x)|}{e^x} = \infty \quad \text{og} \quad \lim_{x\to\infty} \frac{|P(x)|}{e^x} = 0$$

La $f(x) = \frac{|P(x)|}{e^x}$ og $h(x) = 1$.

Vi har

$$\lim_{x\to\infty} \frac{|P(x)|}{e^x} - 1 = -1$$

og $\lim_{x\to-\infty} \frac{|P(x)|}{e^x} - 1 = \infty$,

dvs. differansen $f(x) - h(x)$ antar både positive og negative verdier på $\mathbb{R}$. Siden $f-h$ er kont. fins c slik at $f(c)-h(c)=0$ ved skjæringssetningen, dvs.

$$f(c) = h(c)$$

$$\frac{|P(c)|}{e^c} = 1$$

$$|P(c)| = e^c. \quad \blacksquare$$