

Seksjon 3.2 Andre punktmetoder

Punktmetoden basert på $a-h, a, a+h$

Newtonformen til interpolerende polynom:

$$p_2(x) = c_0 + c_1(x - (a-h)) + c_2(x - (a-h))(x - a)$$

$$f(a-h) = c_0 \Rightarrow c_0 = f(a-h)$$

$$f(a) = c_0 + c_1 h \Rightarrow c_1 = \frac{f(a) - c_0}{h} = \frac{f(a) - f(a-h)}{h}$$

$$\begin{aligned} f(a+h) &= c_0 + c_1 2h + c_2 2h^2 \Rightarrow c_2 = \frac{f(a+h) - c_0 - 2hc_1}{2h^2} \\ &= \frac{f(a+h) - f(a-h) - 2f(a) + 2f(a-h)}{2h^2} \\ &= \frac{f(a+h) - 2f(a) + f(a-h)}{2h^2} \end{aligned}$$

$$p_2(x) = f(a-h) + \frac{f(a) - f(a-h)}{h}(x - (a-h)) + \frac{f(a+h) - 2f(a) + f(a-h)}{2h^2} \cdot (x - (a-h))(x - a)$$

$$p_2'(x) = \frac{f(a) - f(a-h)}{h} + \frac{f(a+h) - 2f(a) + f(a-h)}{2h^2} ((x-a) + (x - (a-h)))$$

$$\begin{aligned} p_2'(a) &= \frac{f(a) - f(a-h)}{h} + \frac{f(a+h) - 2f(a) + f(a-h)}{2h^2} h \\ &= \frac{2f(a) - 2f(a-h) + f(a+h) - 2f(a) + f(a-h)}{2h} \\ &= \frac{f(a+h) - f(a-h)}{2h} \end{aligned}$$

Dette er stigningstallet til sekanten gjennom $a-h$ og $a+h$, og kalles den symmetriske newtonkvotienten.

Oppgave 3.2.4

a) symmetrisk newton for

$$\begin{aligned} 1. f(x) = x^2: \quad \frac{(a+h)^2 - (a-h)^2}{2h} &= \frac{a^2 + 2ah + \cancel{h^2} - a^2 + 2ah - \cancel{h^2}}{2h} \\ &= \frac{4ah}{2h} = 2a = f'(a) \end{aligned}$$

$$2. f(x) = x : \frac{(a+h) - (a-h)}{2h} = \frac{2h}{2h} = 1 = f'(a)$$

$$3. f(x) = 1 : \frac{1-1}{2h} = 0 = f'(a)$$

b) Gjennomsnittet av $f_r(a) = \frac{f(a+h) - f(a)}{h}$ og $f_l(a) = \frac{f(a) - f(a-h)}{h}$

$$\frac{1}{2} \left(\frac{f(a+h) - f(a)}{h} + \frac{f(a) - f(a-h)}{h} \right) = \frac{f(a+h) - f(a-h)}{2h},$$

som er symmetrisk newton.

Trunkeringsfeilen: Hvis f er 3 ganger kont. deriverbar, så gir Taylors formel med restledd:

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f'''(c_1)}{3!}h^3 \quad c_1 \in [a, a+h]$$

$$f(a-h) = f(a) - f'(a)h + \frac{f''(a)}{2!}h^2 - \frac{f'''(c_2)}{3!}h^3 \quad c_2 \in [a-h, a]$$

$$\left(f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(c)}{3!}(x-a)^3 \right) c \in [a, x].$$

Trekk disse fra hverandre:

$$f(a+h) - f(a-h) = 2f'(a)h + \left(\frac{f'''(c_1)}{6} + \frac{f'''(c_2)}{6} \right) h^3$$

$$\frac{f(a+h) - f(a-h)}{2h} - f'(a) = \frac{1}{12} h^2 (f'''(c_1) + f'''(c_2))$$

Ta absoluttverdi:

$$|E(f; a, h)| = \frac{1}{12} h^2 |f'''(c_1) + f'''(c_2)| \quad M_3 = \max_x |f'''(x)|$$

$$\leq \frac{1}{12} h^2 (|f'''(c_1)| + |f'''(c_2)|)$$

$$\leq \frac{1}{12} h^2 (M_3 + M_3) = \frac{1}{6} h^2 M_3$$

Ta logaritmer: $\log\left(\frac{1}{6} M_3\right) + \log h^2$ ($h = 10^{-k}$)

$$= \log\left(\frac{1}{6} M_3\right) - 2k$$

gjør at feil antar raske

En punktmetode for den andrederiverte, basert på $a-h, a, a+h$.

Vi fant over:

$$p_2'(x) = \frac{f(a) - f(a-h)}{h} + \frac{f(a+h) - 2f(a) + f(a-h)}{2h^2} ((x-a-h) + (x-a))$$

$$p_2''(x) = \frac{f(a+h) - 2f(a) + f(a-h)}{2h^2}$$

$$p_2''(a) = \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}, \text{ som er vår nye punktmetode.}$$

Trunkeringsfeilen: Hvis f er 4 ganger kont. deriverbar, så gir Taylors formel med restledd: $c_i \in [a, a+h]$

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2}h^2 + \frac{f'''(a)}{6}h^3 + \frac{f^{(4)}(c_1)}{24}h^4$$

$$f(a-h) = f(a) - f'(a)h + \frac{f''(a)}{2}h^2 - \frac{f'''(a)}{6}h^3 + \frac{f^{(4)}(c_2)}{24}h^4$$

Legg sammen disse:

$$f(a+h) + f(a-h) = 2f(a) + f''(a)h^2 + \frac{h^4}{24} (f^{(4)}(c_1) + f^{(4)}(c_2))$$

$$\frac{f(a+h) - 2f(a) + f(a-h)}{h^2} - f''(a) = \frac{h^2}{24} (f^{(4)}(c_1) + f^{(4)}(c_2))$$

Dette sier at

$$|E_2(f; a, h)| \leq \frac{h^2}{24} (|f^{(4)}(c_1)| + |f^{(4)}(c_2)|)$$
$$\leq \frac{h^2}{24} (M_4 + M_4) = \frac{h^2}{12} M_4.$$

En 4-punktsmetode basert på $a-2h, a-h, a, a+h, a+2h$

Mer regning å finne p_n .

$$f(a+2h) = f(a) + 2f'(a)h + \dots + \frac{f^{(5)}(c_1)}{120}(2h)^5$$

$$f(a+h) = f(a) + f'(a)h + \dots + \frac{f^{(5)}(c_2)}{120}h^5$$

$$f(a-h) = f(a) - f'(a)h + \dots$$

$$f(a-2h) = f(a) - 2f'(a)h + \dots$$

Legg sammen disse slike bidrag fra $f''(a)$, $f'''(a)$, $f^{(4)}(a)$ kansellerer. Dette gir en metode med trunkeringsfeil $\leq \frac{M_5}{18} h^4$

Oppsummering: De fire metodene vi har sett på har trunkeringsfeil

1. $\leq \frac{h}{2} M_2$ for newtonkvotienten
2. $\leq \frac{h^2}{6} M_3$ for symmetrisk newton
3. $\leq \frac{h^4}{18} M_5$ for 4-punktsmetoden
4. $\leq \frac{M_4}{12} h^2$ for metoden for den andrederiverte.

La oss se på tilfellet $n=2$, $m_0=0$, $m_1=-1$, $m_2=1$

$$V = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad V^{-1} = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = (V^{-1})^T$$

$$V^{-T} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -V^{-T} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & V^{-T} \end{pmatrix} = \dots \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Dette gir:

$$\begin{pmatrix} h p_2'(a) / 1! \\ h^2 p_2''(a) / 2! \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} f(a) \\ f(a-h) \\ f(a+h) \end{pmatrix}$$

$$p_2'(a) = \frac{f(a+h) - f(a-h)}{2h}$$

$$p_2''(a) = \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

} har utledet for
hånd, ved å
regne ut
interpolerende polynom

Oppgave 3.25

til den deriverte.

a) Tilnærming som bruker $f(a-h)$, $f(a)$, $f(a+2h)$.