

Prøveeksamen

Oppgave 1: vis $\underbrace{2^0 + 2^1 + \dots + 2^n}_{VS} = \underbrace{2^{n+1} - 1}_{HS} \quad (n \geq 1)$

$$n=1 : VS : 2^0 + 2^1 = 3$$

$$HS : 2^2 - 1 = 3$$

Anta påstanden er sann for $(1, \dots, n)$. Da er

$$VS : \underbrace{2^0 + \dots + 2^n}_{2^{n+1} - 1} + 2^{n+1} = 2^{n+1} - 1 + 2^{n+1} = 2 \cdot 2^{n+1} - 1 = 2^{(n+1)+1} - 1$$

$$HS : 2^{(n+1)+1} - 1$$

ser at VS og HS blir like, slik at påstanden er sann også for $n+1$. Den er da sann for alle $n \geq 1$.

Oppgave 2 Definer $A = (\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n)$

radreduser, vi kan ha maks m pivotsøyler siden $m < n$.

En søyle må da ikke være en pivotsøyle

Da har vi en fri variabel, og da har systemet $A\vec{x} = \vec{0}$ uendelig mange løsninger. spesielt finnes $\vec{x} \neq \vec{0}$ s.o. $A\vec{x} = \vec{0}$.

Da er $A\vec{x} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \dots + x_n \vec{a}_n = \vec{0}$, slik at $\vec{a}_1, \dots, \vec{a}_n$ er lineært avhengige.



Oppgave 3 Anta motsatt at f ikke er injektiv

Da finnes $x, y \in A$ slik at $f(x) = f(y)$, $x \neq y$.

$$\text{La } V_f = \{f(x) : x \in A\}$$

siden f surjektiv: $V_f = B$, slik at $\#V_f = n$

Vi har også at $V_f = C \cup D$ der

$$C = \{f(x), f(y)\} \Rightarrow \#C = 1$$

$$D = \{f(z), z \in A \setminus \{x, y\}\} \Rightarrow \#D \leq n-2$$

Vi får:

$$n = \#V_f \leq \#C + \#D \leq 1 + (n-2) = n-1$$

$n \leq n-1$ er en motsigelse.

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Oppgave 4

a) $A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 4 \\ 1 & -2 & -6 \end{pmatrix}$. Hva blir A^{-1} ?

Løsning: Vi radreduserer:

$$\begin{pmatrix} 2 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 1 & -2 & -6 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{I \leftrightarrow III} \begin{pmatrix} 1 & -2 & -6 & 0 & 0 & 1 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 2 & -1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{III - 2I} \begin{pmatrix} 1 & -2 & -6 & 0 & 0 & 1 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 3 & 13 & 1 & 0 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \xrightarrow{III - 3II} \begin{pmatrix} 1 & -2 & -6 & 0 & 0 & 1 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -3 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \xrightarrow{II - 4III} \begin{pmatrix} 1 & -2 & -6 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 & 13 & 8 \\ 0 & 0 & 1 & 1 & -3 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{I + 6III} \sim \begin{pmatrix} 1 & -2 & 0 & 6 & -18 & -11 \\ 0 & 1 & 0 & -4 & 13 & 8 \\ 0 & 0 & 1 & 1 & -3 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{I + 2II} \sim \begin{pmatrix} 1 & 0 & 0 & -2 & 8 & 5 \\ 0 & 1 & 0 & -4 & 13 & 8 \\ 0 & 0 & 1 & 1 & -3 & -2 \end{pmatrix}$$

Derfor er $A^{-1} = \begin{pmatrix} -2 & 8 & 5 \\ -4 & 13 & 8 \\ 1 & -3 & -2 \end{pmatrix}$

b) $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

endrer fortegn

endrer ikke determinant.

determinanten blir derfor -1.

Kontroll: $\begin{vmatrix} 2 & -1 & 1 \\ 0 & 1 & 4 \\ 1 & -2 & -6 \end{vmatrix} = 2 \begin{vmatrix} 1 & 4 \\ -2 & -6 \end{vmatrix} + 1 \begin{vmatrix} -1 & 1 \\ 1 & 4 \end{vmatrix} = 2(-6+8) - 4 - 1 = 4 - 5 = -1.$

Oppgave 5

a) $A = \begin{pmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{pmatrix}$. Hva er egenverdier / egenvektorer

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 0.2 & -0.8 \\ -0.8 & \lambda - 0.2 \end{vmatrix} = (\lambda - 0.2)^2 - 0.64$$

Detto er 0 når $\lambda - 0.2 = \pm 0.8 \Rightarrow \lambda_1 = -0.6, \lambda_2 = 1$

Egenvektor for $\lambda_1 = -0.6$:

$$-0.6 I - A = \begin{pmatrix} -0.8 & -0.8 \\ -0.8 & -0.8 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad x_1 + x_2 = 0 \Leftrightarrow x_1 = -x_2$$

$$\text{egenvektorer: } \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ spesiell egenvektor} \right)$$

Egenvektor for $\lambda_2 = 1$:

$$I - A = \begin{pmatrix} 0.8 & -0.8 \\ -0.8 & 0.8 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad x_1 - x_2 = 0 \Rightarrow x_1 = x_2$$

$$\text{egenvektorer: } \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ spesiell egenvektor} \right)$$

b) Hva blir $k(A)$: $k(A) = \left| \frac{\lambda_2}{\lambda_1} \right| = \frac{1}{1-0.6} = \frac{1}{0.6} = \frac{5}{3}$.

NB! A er symmetrisk, slik at $k(A)$ er definert.

Oppgave 6 $f(x) = x^4 - x^3$

Vi skal finne $p_3(x)$, som interpolerer f i $-1, 0, 1, 2$:

$$f(-1) = 2, \quad f(0) = 0, \quad f(1) = 0, \quad f(2) = 8$$

$$p_3(x) = c_0 + c_1(x+1) + c_2(x+1)x + c_3(x+1)x(x-1)$$

$$(x=-1): 2 = c_0 \quad \Rightarrow c_0 = 2$$

$$(x=0): 0 = c_0 + c_1 \quad \Rightarrow c_1 = -2$$

$$(x=1): 0 = c_0 + 2c_1 + 2c_2 \quad \Rightarrow c_2 = \frac{0-2+4}{2} = 1$$

$$(x=2): 8 = c_0 + 3c_1 + 6c_2 + 6c_3 \Rightarrow c_3 = \frac{8-2+6-6}{6} = 1$$

$$p_3(x) = 2 - 2(x+1) + (x+1)x + (x+1)x(x-1)$$

Oppgave 7 $f''(a) \sim \frac{f(a-h) - 2f(a) + f(a+h)}{h^2}$

feilen i tilnærmingen for $f(x) = x^4$?

$$f''(x) = 12x^2 \Rightarrow f''(a) = 12a^2$$

finnøring:
$$\frac{(a-h)^4 - 2a^4 + (a+h)^4}{h^2}$$

$$= \frac{(\cancel{a^4} - \cancel{4a^3h} + 6a^2h^2 - \cancel{4ah^3} + \cancel{h^4}) - \cancel{2a^4} + (\cancel{a^4} + \cancel{4a^3h} + 6a^2h^2 + \cancel{4ah^3} + \cancel{h^4})}{h^2}$$

$$= \frac{12a^2h^2 + 2h^4}{h^2} = 12a^2 + 2h^2$$

Feilen blir: $|12a^2 - (12a^2 + 2h^2)| = \underline{\underline{2h^2}}$

Oppgave 8: $\int_1^9 \cos(\pi x / 12) dx$, midtpunktsmetoden, 4 intervaller:

Løsning: $h = \frac{b-a}{n} = \frac{9-1}{4} = 2$

intervallene: $[1, 3]$, $[3, 5]$, $[5, 7]$, $[7, 9]$

midtpunkter: 2 , 4 , 6 , 8

$$h (f(2) + f(4) + f(6) + f(8))$$

$$= 2 \left(\cos\left(\frac{2\pi}{12}\right) + \cos\left(\frac{4\pi}{12}\right) + \cos\left(\frac{6\pi}{12}\right) + \cos\left(\frac{8\pi}{12}\right) \right)$$

$$= 2 \left(\cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{2\pi}{3}\right) \right)$$

$$= 2 \left(\frac{\sqrt{3}}{2} + \cancel{\frac{1}{2}} + 0 - \cancel{\frac{1}{2}} \right)$$

$$= \underline{\underline{\sqrt{3}}}$$