

Fra sist gang

Newtonformen for det interpolerende polynomiet av grad ≤ 2 :

$$p_2(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)(x-x_1)$$

p_2 er en parabel hvis $c_2 \neq 0$

Interpolasjonsbetingelsene

$$f(x_0) = c_0$$

$$f(x_1) = c_0 + c_1(x_1 - x_0)$$

$\vdots$

$$f(x_n) = c_0 + c_1(x_n - x_0) + \dots + c_n(x_n - x_0) \dots (x_n - x_{n-1})$$

De $k+1$ første likningene bruker bare $(x_0, f(x_0))$,
 $(x_1, f(x_1))$, $\dots$ $(x_k, f(x_k))$, slik at
 c_k bare avhenger av disse.

Vi skriver $c_k = f[x_0, \dots, x_k]$;

$$\begin{aligned} p_n(x) &= \underbrace{f[x_0] + f[x_0, x_1](x-x_0) + \dots}_{p_{n-1}(x)} + \underbrace{f[x_0, \dots, x_n](x-x_0) \dots (x-x_{n-1})}_{c_n} \\ &= p_{n-1}(x) + \underbrace{f[x_0, \dots, x_n]}_{c_n} (x-x_0) \dots (x-x_{n-1}) \end{aligned}$$

Oppgave 5 $f(x) = x^2$. Finn det unike polynomiet av grad ≤ 3
som interpolerer f i $x_0=0, x_1=1, x_2=2, x_3=3$.

kanonisk form: For å finne koeffisienter på kanonisk form

$$c_0 + c_1 x + c_2 x^2 + \dots$$

$$\text{Løs } V\vec{c} = \vec{f}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \textcircled{1} & 1 & 1 & 1 & 1 \\ \textcircled{1} & 2 & 4 & 8 & 4 \\ \textcircled{1} & 3 & 9 & 27 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & \textcircled{2} & 4 & 8 & 4 \\ 0 & \textcircled{3} & 9 & 27 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 & 2 \\ 0 & 0 & 6 & 24 & 6 \end{pmatrix}$$

V F

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & \textcircled{3} & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \textcircled{1} & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

sier af

$$\begin{aligned} c_0 &= 0 \\ c_1 &= 0 \\ c_2 &= 1 \\ c_3 &= 0 \end{aligned}$$

p8 kanonisk form: $c_0 + c_1 x + c_2 x^2 + c_3 x^3$

$$= x^2$$

Newtonformen:

$$f(x_0) = 0 = c_0$$

$$f(x_1) = 1 = c_0 + \overset{x_1 - x_0}{(1-0)} c_1$$

$$f(x_2) = 4 = c_0 + (2-0)c_1 + \overset{x_2 - x_0}{(2-0)} \overset{x_2 - x_1}{(2-1)} c_2$$

$$f(x_3) = 9 = c_0 + (3-0)c_1 + (3-0) \overset{x_3 - x_0}{(3-1)} \overset{x_3 - x_1}{(3-2)} c_2 + (3-0) \overset{x_3 - x_0}{(3-1)} \overset{x_3 - x_1}{(3-2)} \overset{x_3 - x_2}{(3-3)} c_3$$

$$0 = c_0$$

$$1 = c_0 + c_1$$

$$4 = c_0 + 2c_1 + 2c_2$$

$$9 = c_0 + 3c_1 + 6c_2 + 6c_3$$

$$c_0 = 0$$

$$c_1 = 1 - c_0 = 1$$

$$c_2 = \frac{4 - c_0 - 2c_1}{2} = \frac{4 - 2}{2} = 1$$

$$c_3 = \frac{9 - c_0 - 3c_1 - 6c_2}{6} = \frac{9 - 3 - 6}{6} = 0$$

$$p_3(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + c_3(x - x_0)(x - x_1)(x - x_2)$$

$$= x + x(x-1)$$

$$= x + x(x-1)$$

$$(= x + x^2 - x = x^2)$$

Eksempel 10.6 Rungefunktjonen, $f(x) = \frac{1}{1+x^2}$ på $[-5, 5]$

Teorem 10.7 La $x_0, \dots, x_n \in [a, b]$ være interpolasjonspunkter.

Ant $f: [a, b] \rightarrow \mathbb{R}$ er minst $n+1$ ganger kontinuerlig deriverbar, og p_n det unike interpolerende polynomiet av grad $\leq n$. Da vil

$$|f(x) - p_n(x)| \leq \frac{M_{n+1} (b-a)^{n+1}}{(n+1)!} \text{ for alle } x \in [a, b]$$

$$\max_{x \in [a, b]} |f^{(n+1)}(x)|$$

Altså: Når M_{n+1} ikke vokser for fort, så vil interpolasjonsfeilen være liten.

For Rungefunktjonen: $M_{n+1} \approx (n+1)!$ s.a.

$$|f(x) - p_n(x)| \leq \frac{(n+1)!}{(n+1)!} (b-a)^{n+1} = 10^{n+1},$$

som er et stort tall.

Oppgave 6

x	0	1	3	4
$f(x)$	1	0	2	1

(a) Lagrangeformen til interpolerende polynom:

$$\begin{aligned}
 p_3(x) &= c_0 (x-1)(x-3)(x-4) & x=0 \rightarrow c_0 (-1)(-3)(-4) &= \underline{-12c_0} \\
 &+ c_1 \cdot x (x-3)(x-4) & x=1 \rightarrow c_1 \cdot 1(-2)(-3) &= \underline{6c_1} \\
 &+ c_2 \cdot x (x-1)(x-4) & x=3 \rightarrow c_2 \cdot 3 \cdot 2 \cdot (-1) &= \underline{-6c_2} \\
 &+ c_3 \cdot x (x-1)(x-3) & x=4 \rightarrow c_3 \cdot 4 \cdot 3 \cdot 1 &= \underline{12c_3}
 \end{aligned}$$

Setter inn interpolasjonspunkter:

$$\left. \begin{aligned}
 1 &= -12c_0 \\
 0 &= 6c_1 \\
 2 &= -6c_2 \\
 1 &= 12c_3
 \end{aligned} \right\} \begin{aligned}
 c_0 &= -\frac{1}{12} \\
 c_1 &= 0 \\
 c_2 &= -\frac{1}{3} \\
 c_3 &= \frac{1}{12}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow p_3(x) &= -\frac{1}{12} (x-1)(x-3)(x-4) \\
 &\quad -\frac{1}{3} x (x-1)(x-4) \\
 &\quad + \frac{1}{12} x (x-1)(x-3)
 \end{aligned}$$

Newton formen:

$$1 = C_0$$

$$0 = C_0 + C_1 (x_1 - x_0)$$

$$2 = C_0 + C_1 (x_2 - x_0) + C_2 (x_2 - x_0)(x_2 - x_1)$$

$$1 = C_0 + C_1 (x_3 - x_0) + C_2 (x_3 - x_0)(x_3 - x_1) + C_3 (x_3 - x_0)(x_3 - x_1)(x_3 - x_2)$$

4 3 1

$$1 = C_0$$

$$0 = C_0 + C_1$$

$$2 = C_0 + 3C_1 + 6C_2$$

$$1 = C_0 + 4C_1 + 12C_2 + 12C_3$$

$$C_0 = 1$$

$$C_1 = -1$$

$$C_2 = \frac{2 - C_0 - 3C_1}{6} = \frac{2 - 1 + 3}{6} = \frac{2}{3}$$

$$C_3 = \frac{1 - C_0 - 4C_1 - 12C_2}{12} = \frac{1 - 1 + 4 - 8}{12}$$

$$= -\frac{4}{12} = -\frac{1}{3}$$

$$p_3(x) = 1 - x + \frac{2}{3}x(x-1) - \frac{1}{3}x(x-1)(x-3)$$

(c) For å verifisere at vi har regnet riktig :

Sett inn $x = x_i$, i stedet for å gange ut :

punkt	Newton form	Lagrange form	funksjonsverdier
$x=0$	<u>1</u>	$-\frac{1}{12}(-1)(-3)(-4) = \underline{1}$	1
$x=1$	<u>0</u>	<u>0</u>	0
$x=3$	$1 - 3 + \frac{2}{3} \cdot 3 \cdot 2 = \underline{2}$	$-\frac{1}{3} \cdot 3 \cdot 2 \cdot (-1) = \underline{2}$	2
$x=4$	$1 - 4 + \frac{2}{3} \cdot 12 - \frac{1}{3} \cdot 12$ $= -3 + 8 - 4$ $= \underline{1}$	$\frac{1}{12} \cdot 4 \cdot 3 \cdot 1 = \underline{1}$	1