

Punktmetode for den andredederiverte,

basert på samme punkter ($f(a-h)$, $f(a)$, $f(a+h)$)

Vi hadde:

$$p_2(x) = f(a-h) + \frac{f(a) - f(a-h)}{h} (x - (a-h)) + \frac{f(a+h) - 2f(a) + f(a-h)}{2h^2} \underbrace{(x - (a-h))(x - a)}_{\substack{x^2 - ax - (a-h)x + a(a-h) \\ \frac{x^2}{2} - ax - \frac{(a-h)x}{2} + a(a-h)}}$$

$$p_2''(a) = \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

Trunkeringsfeil: ($p_2''(a) - f''(a)$)

$$f(a+h) = f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + \frac{1}{6}f'''(a)h^3 + \frac{1}{24}f^{(4)}(c_1)h^4$$

$$f(a-h) = f(a) - f'(a)h + \frac{1}{2}f''(a)h^2 - \frac{1}{6}f'''(a)h^3 + \frac{1}{24}f^{(4)}(c_2)h^4$$

Legg sammen:

$$f(a+h) + f(a-h) = 2f(a) + f''(a)h^2 + \frac{1}{24}f^{(4)}(c_1)h^4 + \frac{1}{24}f^{(4)}(c_2)h^4$$

Flytt over, del med h^2

$$\frac{f(a+h) - 2f(a) + f(a-h)}{h^2} - f''(a) = \frac{1}{24}f^{(4)}(c_1)h^2 + \frac{1}{24}f^{(4)}(c_2)h^2$$

$p_2''(a)$

$$p_2''(a) - f''(a) = E_2(f; a, h)$$

Tar absolutthverdi:

$$|E_2(f; a, h)| = \left| \frac{1}{24}f^{(4)}(c_1)h^2 + \frac{1}{24}f^{(4)}(c_2)h^2 \right|$$

$$\leq \frac{h^2}{24} (|f^{(4)}(c_1)| + |f^{(4)}(c_2)|)$$

$$\leq \frac{h^2}{24} (M_4 + M_4) = \frac{h^2}{12} M_4$$

Oppsummering Trunkeringsferl:

Newtonkvotienten:

$$|E_1(f; a, h)| \leq \frac{h}{2} M_2$$

Symmetrisk Newton:

$$|E_s(f; a, h)| \leq \frac{h^2}{6} M_3$$

Andrederiverte:

$$|E_2(f; a, h)| \leq \frac{h^2}{12} M_4$$

stign.tall : $\log \log$:

2 2 1

Oppgave 11.8

a) Tilnærming til $f'(a)$ som bare bruker $f(a-h), f(a), f(a+2h)$ interpolerende polynom:

$$C_0 = f(x_0) = f(a-h)$$

$$C_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(a) - f(a-h)}{h} \quad \left. \vphantom{\frac{f(a) - f(a-h)}{h}} \right\} \text{som før.}$$

$$C_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$= \frac{\frac{f(a+2h) - f(a)}{2h} - \frac{f(a) - f(a-h)}{h}}{3h}$$

$$= \frac{f(a+2h) - f(a) - 2f(a) + 2f(a-h)}{6h^2}$$

$$= \frac{f(a+2h) - 3f(a) + 2f(a-h)}{6h^2}$$

$$P_2(x) = f(a-h) + \frac{f(a) - f(a-h)}{h} (x - (a-h)) \\ + \frac{f(a+2h) - 3f(a) + 2f(a-h)}{6h^2} (x - (a-h))(x - a)$$

$$P_2'(x) = \frac{f(a) - f(a-h)}{h} + \frac{f(a+2h) - 3f(a) + 2f(a-h)}{6h^2} (\underbrace{x - (a-h) + x - a}_{a - (a-h) + a - a = h})$$

$$P_2'(a) = \frac{f(a) - f(a-h)}{h} + \frac{f(a+2h) - 3f(a) + 2f(a-h)}{6h}$$

$$= \frac{f(a+2h) + 3f(a) - 4f(a-h)}{6h}$$

$$|f'''(c)| \leq M_3$$

Trunkeringsfeilen

$$f(a-h) = f(a) - f'(a)h + \frac{f''(a)}{2}h^2 - \frac{f'''(c)}{6}h^3 \quad (1)$$

$$f(a+2h) = f(a) + 2f'(a)h + 2f''(a)h^2 + \frac{f'''(c)}{6}8h^3 \quad (2)$$

gang (1) med 4, trekk fra (2):

$$\frac{1}{6} + \frac{8}{6} = \dots$$

$$4f(a-h) - f(a+2h) = 3f(a) - 6f'(a)h + CM_3 h^3$$

$$f'(a) - \frac{f(a+2h) + 3f(a) - 4f(a-h)}{6h} = CM_3 h^2$$

Trunkeringsfeil $\leq CM_3 h^2$, der C er en konstant.

For hvilke polynomer er Newtonkvotienten eksakt?

$$E_1(f; a, h) = 0 \quad \text{hvis} \quad M_2 = 0, \quad \left(\frac{h}{2} M_2 \right)$$

Detto skjer når $f''(x) = 0$, dvs. for alle lineære funksjoner $f(x) = bx + c$

Sjekk: $\frac{f(a+h) - f(a)}{h} = \frac{b(a+h) + c - (ba + c)}{h} = \frac{\cancel{ba} + bh + \cancel{c} - \cancel{ba} - \cancel{c}}{h}$

$= b = f'(a)$, så metoden er eksakt for slik f .

Men for $f(x) = x^2$:

$$\frac{f(a+h) - f(a)}{h} = \frac{(a+h)^2 - a^2}{h} = \frac{\cancel{a^2} + 2ah + \cancel{h^2} - \cancel{a^2}}{h} = 2a + h$$

$\neq f'(a) = 2a$ ↑
feil.

For hvilke polynomer er symmetrisk Newton eksakt?

$E(f; a, h) = 0$ hvis $M_3 = 0$. Dette skjer når $f'''(x) = 0$ for alle x ,

dvs for alle andregradspolynomer.

Sjekk:

$$f(x) = x^2: \quad \frac{f(a+h) - f(a-h)}{2h} = \frac{(a+h)^2 - (a-h)^2}{2h}$$
$$= \frac{\cancel{a^2} + 2ah + \cancel{h^2} - \cancel{a^2} + 2ah - \cancel{h^2}}{2h} = \frac{4ah}{2h} = 2a = f'(a)$$

så metoden er eksakt for slik f .

Men: $f(x) = x^3$:

$$\frac{f(a+h) - f(a-h)}{2h} = \frac{(a+h)^3 - (a-h)^3}{2h}$$
$$= \frac{\cancel{a^3} + 3a^2h + 3a\cancel{h^2} + \cancel{h^3} - \cancel{a^3} + 3a^2h - 3a\cancel{h^2} + \cancel{h^3}}{2h}$$
$$= \frac{6a^2h + 2h^3}{2h} = 3a^2 + h^2 \neq f'(a) = 3a^2$$

så metoden er ikke eksakt for $f(x) = x^3$.

Vise direkte at symmetrisk Newton er eksakt for alle andregradspolynomer:

$$f(x) = x: \quad \frac{f(a+h) - f(a-h)}{2h} = \frac{\cancel{a+h} - \cancel{(a-h)}}{2h} = \frac{2h}{2h} = 1 = f'(a)$$

$$f(x) = 1: \quad \frac{f(a+h) - f(a-h)}{2h} = \frac{1 - 1}{2h} = 0 = f'(a)$$

Siden metoden er eksakt for $f(x) = 1, x, x^2$, så er den også eksakt for alle andregradspolynomer:

Sett $f(x) = b + cx + dx^2$:

$$\frac{f(a+h) - f(a-h)}{2h} = \frac{b + c(a+h) + d(a+h)^2 - b - c(a-h) - d(a-h)^2}{2h}$$

$$= b \frac{1-1}{2h} + c \frac{(a+h) - (a-h)}{2h} + d \frac{(a+h)^2 - (a-h)^2}{2h}$$

= 1 siden metoden
er eksakt for
 $f(x) = x$

= 2a siden metoden
er eksakt for
 $f(x) = x^2$

$$= c + 2da = f'(a).$$

Merk: Newtonkvotienten er ikke eksakt for noe andregradspolynom, men f.eks. for tredjegradspolynomet $f(x) = x^3 - 4x^2$ så er den eksakt for $a = h = 1$:

Newtonkvotient

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{2^3 - 4 \cdot 2^2 - 1^3 + 4}{1} \\ &= \frac{8 - 16 - 1 + 4}{1} = -5 \end{aligned}$$

derivasjon

$$\begin{aligned} f'(x) &= 3x^2 - 8x \\ f'(a) &= f'(1) = 3 - 8 = -5 \end{aligned}$$