

Foils for Chapter 3 in linear algebra and numerical methods

Øyvind Ryan

2026-09-18

Gaussian elimination

A general system with m equations and n unknowns:

$$\begin{array}{cccccc} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & = & b_1 \\ a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n & = & b_m \end{array} \quad (1)$$

Chapter 2. we wrote this more compactly as $A\mathbf{x} = \mathbf{b}$,

der

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Procedure in Gaussian elimination

The system $A\mathbf{x} = \mathbf{b}$ is solved by systematically eliminating variables from left to right. Suppose that the variables $x_1, \dots, x_n$ have been eliminated. There are two possibilities:

1. x_{n+1} does not appear in the remaining equations. Then x_{n+1} has been eliminated as well, and we proceed with the elimination of x_{n+2} .
2. x_{n+1} appears in at least one of the remaining equations. We do the following:
 - (a) Take one of the equations where x_{n+1} appears, and place this first among the remaining equations.
 - (b) Scale this equation so that there is a 1 in front of x_{n+1} . We shall call this a *leading one* (there should be no contribution left of x_{n+1} in this equation).
 - (c) Add multiples of this equation to the other remaining equations so that x_{n+1} gets eliminated.

We have now eliminated x_{n+1} , and proceed with the elimination of x_{n+2} .

A system with a unique solution

$$\begin{array}{rrcr} & -8x_2 & +3x_3 & = 6 \\ 4x_1 & -3x_2 & +2x_3 & = 8 \\ \rightarrow x_1 & -2x_2 & +x_3 & = 3 \end{array}$$

bytt om
får ledende enen
i rad 1

$$\begin{aligned} (-4)(-2) - 3 \\ = 8 - 3 = 5 \end{aligned}$$

$$\begin{array}{rrcr} x_1 & -2x_2 & +x_3 & = 3 \\ \textcircled{4x_1} & -3x_2 & +2x_3 & = 8 \\ & -8x_2 & +3x_3 & = 6 \end{array}$$

legg til
-4 • likning 1
til likning 2

$$\begin{array}{rrcr} x_1 & -2x_2 & +x_3 & = 3 \\ & \textcircled{5x_2} & -2x_3 & = -4 \\ & -8x_2 & +3x_3 & = 6 \end{array}$$

få ledende enen
i rad 2.
Gang likning 2 med $\frac{1}{5}$

$$\begin{array}{rrcr} x_1 & -2x_2 & +x_3 & = 3 \\ & x_2 & -\frac{2}{5}x_3 & = -\frac{4}{5} \\ & \textcircled{-8x_2} & +3x_3 & = 6 \end{array}$$

legg til 8 • likning 2
i likning 3

$$3 - \frac{16}{5} = -\frac{1}{5}$$

$$-\frac{32}{5} + 6$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 3 \\ x_2 - \frac{2}{3}x_3 &= -\frac{4}{5} \\ -\frac{1}{5}x_3 &= -\frac{2}{5} \end{aligned}$$

gang likning 3
med -5

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 3 \\ x_2 - \frac{2}{5}x_3 &= -\frac{4}{5} \\ x_3 &= 2 \end{aligned}$$

trekk fra likning
3 i ligning 1.

legg til $\frac{2}{5} \cdot$ likning 3
i likning 2

Systemet er nå på trappeform:

1. Hver likning er enten $0=0$ eller har en ledende ener
2. Hver likning begynner med minst en null mer enn
likningen over.

Ferdig med å eliminere variable fra venstre mot høyre.

Fortsetter med å eliminere variable fra høyre mot venstre:

$$\begin{aligned}x_1 - 2x_2 &= 1 \\x_2 &= 0 \\x_3 &= 2\end{aligned}$$

Legg til 2 · likning 2
i likning 1

$$\begin{aligned}x_1 &= 1 \\x_2 &= 0 \\x_3 &= 2\end{aligned}$$

Vi fikk en unik løsning

Systemet er nå på redusert trappeform:

1. Det er på trappeform
2. Hver variabel med en ledende ener forekommer bare i en likning.

Disse kalles også for pivotvariable

her: x_1, x_2, x_3

posisjoner til en ledende ener kalles også for et pivotelement.

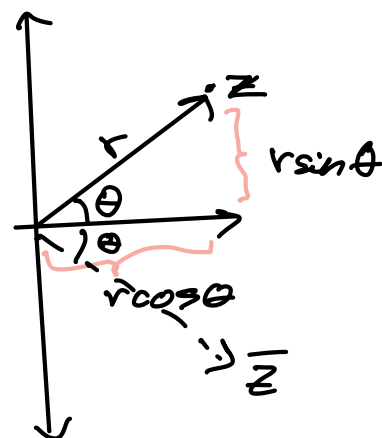
her: pivotelementene er $(1,1)$, $(2,2)$, $(3,3)$

Repetisjon om komplekse tall

$$x = 2 + 3i \quad \leftarrow \text{imaginærenhet}$$

$\downarrow \qquad \downarrow$

real del imaginær del
 $\operatorname{Re} x$ $\operatorname{Im} x$



Konjugasjon: $\overline{2+3i} = 2-3i$

$$i^2 = -1$$

$$(2+3i)(2+4i) = \underline{4} + 8i + 6i + \underline{12 \cdot (-1)}$$
$$= -8 + 14i$$

$$\overline{ab} = \bar{a}\bar{b}, \quad \overline{a+b} = \bar{a} + \bar{b}$$

Lengde: $|2+3i| = \sqrt{2^2 + 3^2} = \sqrt{13}$

Skalarprodukt: $\vec{a} \cdot \vec{b} = a_1 \bar{b}_1 + a_2 \bar{b}_2 + \dots + a_n \bar{b}_n$

$$\vec{a} = \begin{pmatrix} 2+i \\ 1-i \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 3+2i \\ 3+3i \end{pmatrix}$$

$$\begin{aligned} \vec{a} \cdot \vec{b} &= (2+i)(3-2i) + (1-i)(3-3i) \\ &= \underline{6} - 4i + 3i + \underline{2} + \cancel{3} - 3i - 3i - \cancel{3} \\ &= 8 - 7i. \end{aligned}$$

$$a + \bar{a} = 2 \operatorname{Re} a$$

The augmented matrix

We will assemble both A and $\mathbf{b}$ in the bigger matrix $(A \ \mathbf{b})$, called the augmented matrix of the system. For the example above:

utvidet matrise

$$A = \begin{pmatrix} 0 & -8 & 3 \\ 4 & -3 & 2 \\ 1 & -2 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix} \quad (A \ \mathbf{b}) = \begin{pmatrix} 0 & -8 & 3 & 6 \\ 4 & -3 & 2 & 8 \\ 1 & -2 & 1 & 3 \end{pmatrix}$$

- ▶ An equation: a row in the augmented matrix.
- ▶ A variable: a column in the coefficient matrix A .
- ▶ Pivot row: A row with a leading one.
- ▶ Pivot column: A column with a leading one.

Gaussian elimination on matrix form

dropp variable, likhetstegn, +/-. Kun tall + parenteser

$$\begin{aligned} \begin{pmatrix} 0 & -8 & 3 & 6 \\ 4 & -3 & 2 & 8 \\ 1 & -2 & 1 & 3 \end{pmatrix} &\sim \begin{pmatrix} 1 & -2 & 1 & 3 \\ \color{red}{4} & -3 & 2 & 8 \\ 0 & -8 & 3 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 & 3 \\ 0 & 5 & -2 & -4 \\ 0 & -8 & 3 & 6 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & -2 & 1 & 3 \\ 0 & 1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & \color{red}{-8} & 3 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 & 3 \\ 0 & 1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & 0 & -\frac{1}{5} & -\frac{2}{5} \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & -2 & \color{red}{1} & 3 \\ 0 & 1 & \color{red}{-\frac{2}{5}} & -\frac{4}{5} \\ 0 & 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & \color{red}{-2} & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \end{aligned}$$

(2)

$\sim$ denotes one of the operations we performed.

(radekvivalent med)

Gaussian elimination with sympy.

```
C, cols = sym.Matrix([[0,-8,3,6],  
                      [4,-3,2,8],  
                      [1,-2,1,3]]).rref()  
print(C)
```

Matrix([[1, 0, 0, 1], [0, 1, 0, 0], [0, 0, 1, 2]])

The same matrix was obtained. This is not obvious (there are many different ways to bring a matrix to reduced echelon form).

Next week: Yes, the reduced echelon form is actually unique!

Gaussian elimination with the operations indicated

1. The first operation swapped row 1 and 3. Will write this as $I \leftrightarrow III$.
2. The second operation subtracted 4 times row 1 in row 2. Will write this as $II - 4I$.
3. The third operation multiplied row 2 with $1/5$. Will write this as $II \cdot (1/5)$.

$$\begin{pmatrix} 0 & -8 & 3 & 6 \\ 4 & -3 & 2 & 8 \\ 1 & -2 & 1 & 3 \end{pmatrix} \xrightarrow{I \leftrightarrow III} \begin{pmatrix} 1 & -2 & 1 & 3 \\ 4 & -3 & 2 & 8 \\ 0 & -8 & 3 & 6 \end{pmatrix}$$

$$\xrightarrow{II - 4I} \begin{pmatrix} 1 & -2 & 1 & 3 \\ 0 & 5 & -2 & -4 \\ 0 & -8 & 3 & 6 \end{pmatrix} \xrightarrow{II \cdot (1/5)} \begin{pmatrix} 1 & -2 & 1 & 3 \\ 0 & 1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & -8 & 3 & 6 \end{pmatrix}$$

$$\xrightarrow{III + 8II} \begin{pmatrix} 1 & -2 & 1 & 3 \\ 0 & 1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & 0 & -\frac{1}{5} & -\frac{2}{5} \end{pmatrix} \xrightarrow{III \cdot (-5)} \begin{pmatrix} 1 & -2 & 1 & 3 \\ 0 & 1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$II+\frac{2}{5}\underset{\sim}{III},I-III\left(\begin{array}{cccc}1&-2&0&1\\0&1&0&0\\0&0&1&2\end{array}\right)\underset{\sim}{I+2II}\left(\begin{array}{cccc}1&0&0&1\\0&1&0&0\\0&0&1&2\end{array}\right)\quad(3)$$

Elementary row operations

The system was thus brought to reduced echelon form with help of *elementary row operations*:

1. Add a multiple of one row to another,
2. multiply a row with a constant $\neq 0$,
3. Swap two rows.

We say that A and B are *row equivalent* (written $A \sim B$) if we can transform A to B through a series of elementary row operations. From the discussion above:

$$\begin{pmatrix} 0 & -8 & 3 & 6 \\ 4 & -3 & 2 & 8 \\ 1 & -2 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Every matrix is row equivalent to a matrix on reduced echelon form.

Expressing elementary row operations in terms of matrices

Proposition

Suppose B is obtained by applying an elementary row operation to the $m \times n$ -matrix A . Then $B = EA$ where E is the matrix obtained by applying the same row operation to the identity matrix I_m .

A matrix is called *elementary* if it is obtained by applying an elementary row operation to the identity matrix.

Proof: Sett $E = I_m + c \underbrace{\vec{e}_i \vec{e}_j^T}_{\text{matrise med } c \text{ på plass } (i,j)}$ $i \neq j$
 $\underbrace{\hspace{10em}}_{\text{som identitetsmatrisen, men med } E_{ij} = c}$
 E fås fra I_m ved å legge til c ganger rad j til rad i .
La $\vec{x} \in \mathbb{R}^m$: $\vec{x}$ og $E\vec{x}$ er like, untatt i komponent i ,
der $(E\vec{x})_i = x_i + cx_j$

Hvis A er matrise: A og EA er like, untatt i rad i ,
der $(EA)_{i,:} = a_{i,:} + c a_{j,:}$

Altså: EA er matrisen der $c \cdot \text{rad } j$ legges til i rad i ,
der E er matrisen der $c \cdot \text{rad } j$ er lagt til i
rad i i I_m .

Tilsvarende for radbytter, og å gange en rad med en
skalar. ■

Examples of elementary matrices

$$E_1 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (4)$$

- ▶ E_1 corresponds to the row operation where we add 2 times row 2 to row 1,
- ▶ E_2 to the row operation where we multiply row 3 with 3,
- ▶ E_3 to the row operation where we swap rows 2 and 3.

Row operations can be reversed

The reverse row operations of Equation 4 are

- ▶ adding -2 times row 2 to row 1,
- ▶ multiplying row 3 with $1/3$,
- ▶ swapping rows 2 and 3.

Corresponding elementary matrices

$$F_1 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \quad F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

F_i reverses E_i means that

$$F_1 E_1 = I_3 \quad F_2 E_2 = I_3 \quad F_3 E_3 = I_3.$$

Verification

by using Python:

```
import numpy as np
```

```
E_1 = np.array([[1,2,0],[0,1,0],[0,0,1]])
```

```
F_1 = np.array([[1,-2,0],[0,1,0],[0,0,1]])
```

```
E_1 @ F_1
```

```
array([[1, 0, 0],  
       [0, 1, 0],  
       [0, 0, 1]])
```

with multiplication by hand:

$$E, F_1 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 & 1 \cdot (-2) + 2 \cdot 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Factoring a matrix with elementary matrices

Proposition

For every matrix A we can write

$$C = E_k \cdots E_2 E_1 A$$

where C is a matrix on reduced echelon form, and where $E_1, E_2, \dots, E_k$ are elementary matrices.

Bevis: Alle matriser kan bringes til redusert trappeform ved hjelp av radoperasjoner.

La $E_1, E_2, \dots, E_k$ være de tilsvarende elementære matrisene. Da er

$$C = E_k \cdots E_1 A \quad \blacksquare$$

Solutions to a system are unchanged by elementary row operations

Proposition

Assume a system is obtained from another by applying an elementary row operation. Then the two systems have the same solutions.

Proof: $A\vec{x} = \vec{b} \Rightarrow EA\vec{x} = E\vec{b}$,
slik at løsningen av $A\vec{x} = \vec{b}$ også løser
det nye systemet ($EA\vec{x} = E\vec{b}$)

Anta $EA\vec{x} = E\vec{b}$. Gang på begge sider med
matrisen F som svarer til den reverse radoperasjonen:

$$\underbrace{F}_{I_n} EA\vec{x} = \underbrace{FE}_{I_n} \vec{b} \Rightarrow A\vec{x} = \vec{b}$$

slik at $\vec{x}$ løser det gamle systemet også.

A system with infinitely many solutions

$$\begin{array}{rcl} x_1 & +x_2 & +x_3 = 4 \\ 2x_1 & & -x_3 = 2. \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 2 & 0 & -1 & 2 \end{pmatrix} \xrightarrow{\text{II} - 2\text{I}} \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & -2 & -3 & -6 \end{pmatrix} \xrightarrow{\text{II} \cdot (-\frac{1}{2})} \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{3}{2} & 3 \end{pmatrix} \xrightarrow{\text{I} - \text{II}} \begin{pmatrix} 1 & 0 & -\frac{1}{2} & 1 \\ 0 & 1 & \frac{3}{2} & 3 \end{pmatrix}$$

trappeform

reduziert trappeform

serat: $x_1 - \frac{1}{2}x_3 = 1$

$x_2 + \frac{3}{2}x_3 = 3$

$x_1 = \frac{1}{2}x_3 + 1$

$x_2 = -\frac{3}{2}x_3 + 3$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_3 + 1 \\ -\frac{3}{2}x_3 + 3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

We observe:

V_i har en fri variabel (x_3)

V_i har uendelig mange løsninger (en for hvert valg af x_3)

x_1 og x_2 er pivotvariable (kan udtrykkes ved hjælp af den fri variabel).

A system without solutions

$$\begin{array}{rcrcrcrcl} x_1 & & +x_2 & = & 2 \\ 3x_1 & & +3x_2 & = & 5 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 3 & 3 & 5 \end{pmatrix} \xrightarrow{\text{II} - 3\text{I}} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \xrightarrow{\text{II} \cdot (-1)} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \text{ trappform}$$

$$\xrightarrow{\text{I} - 2\text{II}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ redusert trappform.}$$

$$x_1 + x_2 = 0$$

$$0x_1 + 0x_2 = 1$$

We observe:

$0 = 1$, som er umulig, så ingen løsninger.

Siste søyle i $(A \ \vec{b})$ er en pivotsøyle.

Siste rad sier da at $0 = 1$, som er umulig.

The three possibilities in the examples above

A system can have

1. a unique solution (the first example). Here we had only pivot variables, and no pivot element in the last column.
2. infinitely many solutions (the second example). Here we had a free variable, and no pivot element in the last column.
3. no solutions (the third example). Here we had a pivot element in the last column.

We obtained a reduced echelon form in all examples.

These observations hold more generally

Proposition

Any system can be brought to reduced echelon form, and we have the following possibilities:

- 1. The system has no solutions. In this case the last column is a pivot column,*
- 2. The system has solutions. In this case the last column is not a pivot column and we have that*
 - (a) when there are free variables the system has infinitely many solutions (any value of the free variables give a solution),*
 - (b) when there are no free variables (all columns except the last is a pivot column) the system has a unique solution.*

Invertible matrix theorem, first version

Theorem

The following are equivalent for an $n \times n$ -matrix A :

1. A is row equivalent with I_n .
2. $A\mathbf{x} = \mathbf{0}$ has $\mathbf{x} = \mathbf{0}$ as the only solution.
3. $A\mathbf{x} = \mathbf{b}$ has a unique solution for all right hand sides $\mathbf{b}$.
4. A can be written as the product of elementary matrices.

This result will be extended later.

identitetsmatrisen

Proof: $1 \Rightarrow 3$: Anta A kan radreduseres til I_n

$\forall$ i løser $A\vec{x} = \vec{b}$ ved å radredusere $(A \ \vec{b})$

(samme radoperasjoner)

$$(A \ \vec{b}) \sim \dots \sim \left(\begin{array}{ccc|c} 1 & 0 & \dots & b'_1 \\ 0 & 1 & \dots & b'_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{array} \right) \left(\begin{array}{c} b'_1 \\ b'_2 \\ \vdots \\ b'_n \end{array} \right).$$

dette sier at
$$\begin{array}{l} x_1 = b'_1 \\ x_2 = b'_2 \\ \vdots \end{array}$$

$x_n = b_n$
slik at systemet har en unik løsning (gitt $\vec{b}$)

$3 \Rightarrow 2$: er opplagt

$2 \Rightarrow 1$: Løs $A\vec{x} = \vec{0}$ ved å radredusere $(A \mid \vec{0})$

Siden vi har unik løsning, så har vi ingen frie variable. Men da er alle søylene pivotsøyler

Siden A er kvadratisk så er også alle radene pivotrader
slik at $A \sim I_n$

$1 \Rightarrow 4$ Vi antar at $A \sim I_n$, slik at $I_n = E_k \cdots E_1 A$
(E_i er elementære). La F_i være revers radoperasjon.

$$I_n = E_k \cdots E_1 A$$

$$F_k = E_{k-1} \cdots E_1 A$$

$$F_{k-1} F_k = E_{k-2} \cdots E_1 A$$

$\vdots$

$$F_1 F_2 \cdots F_k = A$$

A kan derfor skrives som produkt av elementære matr.

4 $\Rightarrow$ / Hvis $A = F_1 \cdots F_k$ får vi at (gang opp med E_i)

$$\underbrace{E_k \cdots E_2 E_1}_{\text{elementare}} A = I_n$$

Men da er $A \sim I_n$.

Gaussian elimination and complexity (not in the curriculum)

Suppose we have a system with n equations and n unknowns. How many operations (additions, multiplications, etc.) are needed to solve the system/bring it to reduced echelon form?

Suppose for simplicity that the coefficient matrix A is row equivalent with the identity matrix.

Proposition (Number of operations in Gaussian elimination)

Suppose that $A \sim I_n$. About $2n^3/3$ operations are required to bring A to reduced echelon form.

The diagram shows a matrix in reduced echelon form. The first k rows have leading ones and zeros below them. The remaining rows are all zeros. A red circle highlights the first column of the zero rows, and a red bracket indicates the number of rows k .

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Complexity analysis:

$$k + k \sim 2k$$
$$2k^2$$
$$1 + 2k^2 + 2(k+1)^2 + \dots + 2n^2$$
$$1^2 + 2^2 + 3^2 + \dots + n^2$$
$$\sim n^3$$

Simultaneous solution of systems

Suppose we want to solve two systems $A\mathbf{x} = \mathbf{b}_1$ and $A\mathbf{x} = \mathbf{b}_2$ (same coefficient matrix, different right hand sides).

- ▶ Bring A to reduced echelon form, C ,
- ▶ apply the same row operations to $\mathbf{b}_1$ and $\mathbf{b}_2$, and get $\mathbf{b}'_1$ and $\mathbf{b}'_2$:

$$(A \quad \mathbf{b}_1 \quad \mathbf{b}_2) \sim (C \quad \mathbf{b}'_1 \quad \mathbf{b}'_2)$$

(row reducing A , $\mathbf{b}_1$, and $\mathbf{b}_2$ separately can also be done by row reducing $(A \quad \mathbf{b}_1 \quad \mathbf{b}_2)$ as a whole).

From $(C \quad \mathbf{b}'_1 \quad \mathbf{b}'_2)$ we find the solutions to the two systems.

First example revisited

Our first example solved the system $A\mathbf{x} = \mathbf{b}_1$ where $\mathbf{b}_1 = (6, 8, 3)$. Using the same row operations we can simultaneously solve the system $A\mathbf{x} = \mathbf{b}_2$ where $\mathbf{b}_2 = (1, 8, 2)$:

$$(A \quad \mathbf{b}_1 \quad \mathbf{b}_2) = \begin{pmatrix} 0 & -8 & 3 & 6 & 1 \\ 4 & -3 & 2 & 8 & 8 \\ 1 & -2 & 1 & 3 & 2 \end{pmatrix}$$


$$\stackrel{I \leftrightarrow III}{\sim} \begin{pmatrix} 1 & -2 & 1 & 3 & 2 \\ 4 & -3 & 2 & 8 & 8 \\ 0 & -8 & 3 & 6 & 1 \end{pmatrix}$$

$$\stackrel{II - 4I}{\sim} \begin{pmatrix} 1 & -2 & 1 & 3 & 2 \\ 0 & 5 & -2 & -4 & 0 \\ 0 & -8 & 3 & 6 & 1 \end{pmatrix}$$

$$8 - 8 = 0$$

$$\stackrel{II \cdot (1/5)}{\sim} \begin{pmatrix} 1 & -2 & 1 & 3 & 2 \\ 0 & 1 & -\frac{2}{5} & -\frac{4}{5} & 0 \\ 0 & -8 & 3 & 6 & 1 \end{pmatrix}$$

$$\begin{matrix} III+8II \\ \sim \end{matrix} \begin{pmatrix} 1 & -2 & 1 & 3 & 2 \\ 0 & 1 & -\frac{2}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & -\frac{1}{5} & -\frac{2}{5} & 1 \end{pmatrix} \quad 1 + 8 \cdot 0 = 1$$

$$\begin{matrix} III \cdot (-5) \\ \sim \end{matrix} \begin{pmatrix} 1 & -2 & 1 & 3 & 2 \\ 0 & 1 & -\frac{2}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 & 2 & -5 \end{pmatrix}$$

$$\begin{matrix} II + \frac{2}{5} III, I - III \\ \sim \end{matrix} \begin{pmatrix} 1 & -2 & 0 & 1 & 7 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 2 & -5 \end{pmatrix}$$

$$\begin{matrix} I + 2II \\ \sim \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 2 & -5 \end{pmatrix} \quad 7 + 2 \cdot (-2) = 7 - 4 = 3$$

Thus,

- ▶ $A\mathbf{x}_1 = \mathbf{b}_1$ as before has the solution $\mathbf{x}_1 = (1, 0, 2)$,
- ▶ $A\mathbf{x}_2 = \mathbf{b}_2$ has the solution $\mathbf{x}_2 = (3, -2, -5)$.

Skeleton for Gaussian elimination (not in the curriculum)

```
piv = np.zeros(m).astype(int) # Stores pivot positions
i, j = 0, 0
while i < m and j < n: # Start scan for next pivot
    k = i
    while k < m and abs(A[k,j]) < 10**(-6):
        k += 1
    if k == m: # Need to go to the next column
        j += 1
    else: # Found a nonzero in the current column
        # The next pivot element is in row k. If k>i,
        # swap row i and k.
        # Create a leading one at pos. (i,j)
        # Zero out col. j below row i.
        piv[i] = j
        i += 1; j += 1
```

Oppgave 6 midtveis 2025

$$\begin{pmatrix} 2 & -1 & 3 & 3 \\ 1 & 4 & 0 & 2 \\ 7 & 10 & 6 & a \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 0 & 2 \\ 2 & -1 & 3 & 3 \\ 7 & 10 & 6 & a \end{pmatrix}$$

$$\text{II} - 2\text{I} \sim \begin{pmatrix} 1 & 4 & 0 & 2 \\ 0 & -9 & 3 & -1 \\ 7 & 10 & 6 & a \end{pmatrix}$$

$$\text{III} - 7\text{I} \sim \begin{pmatrix} 1 & 4 & 0 & 2 \\ 0 & -9 & 3 & -1 \\ 0 & -18 & 6 & a-14 \end{pmatrix}$$

$$\text{III} - 2\text{II} \sim \begin{pmatrix} 1 & 4 & 0 & 2 \\ 0 & -9 & 3 & -1 \\ 0 & 0 & 0 & a-12 \end{pmatrix}$$

For $a \neq 12$ er siste søyle en pivotsøyle $\Rightarrow$ ingen løsning.

For $a=12$: systemet har løsning (uendelig mange)

Oppgave 8

$$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & 2 & 14 \\ 3 & 6 & 1 & 15 \end{pmatrix} \xrightarrow[\text{III} - 3\text{II}]{\text{II} - 2\text{I}} \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{pmatrix}$$

$$\text{III} - 2\text{II} \sim \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



Søyle 1 og 3 er pivotsøyer.

Oppgave 10

$$\begin{pmatrix} 2 & 1 & 8 & 16 \\ 3 & 2 & 14 & 25 \\ 4 & 1 & 12 & 30 \end{pmatrix} \xrightarrow{I \cdot (\frac{1}{2})} \begin{pmatrix} 1 & \frac{1}{2} & 4 & 8 \\ 0 & \frac{1}{2} & 2 & 1 \\ 0 & -1 & -4 & -2 \end{pmatrix}$$

II - 3I

III - 4II

$$\sim \begin{pmatrix} 1 & \frac{1}{2} & 4 & 8 \\ 0 & 1 & 4 & 2 \\ 0 & \frac{1}{2} & 2 & 1 \end{pmatrix}$$

$$\text{III} - \frac{1}{2}\text{II} \sim \begin{pmatrix} 1 & \frac{1}{2} & 4 & 8 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

↑
fri variabel

↑
ikke pivotspalte

uendelig mange løsninger.

Oppgave 11

Vi forsøker løse

$$x_1 \underbrace{\begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix}}_{Ax} + x_2 \underbrace{\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}}_{Ay} = \underbrace{\begin{pmatrix} 5 \\ 20 \\ 3 \end{pmatrix}}_{Az}$$

$$\begin{pmatrix} 4 & -1 & 5 \\ 7 & 2 & 20 \\ 3 & -1 & 3 \end{pmatrix} \sim \dots \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \rightarrow x_1 = 2 \\ \rightarrow x_2 = 3 \end{matrix}$$

Slik at $\begin{pmatrix} 5 \\ 20 \\ 3 \end{pmatrix} = 2 \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix} + 3 \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

$$Az = 2Ax + 3Ay$$

$$Az = A(2x + 3y)$$

$$\text{Daher ist } z = 2x + 3y = 2 \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2+9 \\ 8+0 \\ 4-3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 11 \\ 8 \\ 1 \end{pmatrix}}}$$

(alternativ a)