

Foils for Chapter 4 in linear algebra and numerical methods

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The inverse matrix

Let A be square.

- ▶ A system $A\mathbf{x} = \mathbf{b}$ can be solved by Gaussian elimination.
- ▶ If A satisfies 1.-4. in the (first version of the) invertible matrix theorem, we can consider the mapping $\mathbf{b} \rightarrow \mathbf{x}$ which sends a given right hand side to the unique solution $\mathbf{x}$. We show that this is a linear transformation. Its matrix is denoted by A^{-1} , and is called the *inverse* of A :

Ind vise: 1. Hvis $\vec{x}_1, \vec{x}_2$ er de unikke løsninger af $A\vec{x} = \vec{b}_1$

og $A\vec{x} = \vec{b}_2$, så er $\vec{x}_1 + \vec{x}_2$ den unikke løsning

af $A\vec{x} = \vec{b}_1 + \vec{b}_2$

(følger fra at $A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{b}_1 + \vec{b}_2$)

2. Hvis $\vec{x}$ er den unikke løsning af $A\vec{x} = \vec{b}$, så er

$c\vec{x}$ den unikke løsning af $A\vec{x} = c\vec{b}$.

(følger fra at $A(c\vec{x}) = cA\vec{x} = c\vec{b}$)

- ▶ If we have found A^{-1} , we can solve any system $A\mathbf{x} = \mathbf{b}$ simply by computing $\mathbf{x} = A^{-1}\mathbf{b}$.
- ▶ A^{-1} is uniquely defined.
- ▶ Since $A^{-1}A\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$, and $AA^{-1}\mathbf{b} = \mathbf{b}$ for all $\mathbf{b}$, we have that $A^{-1}A = AA^{-1} = I_n$.
- ▶ We say that A is *invertible* when there exists a square matrix B so that $BA = AB = I_n$. Thus: 1.-4. imply that A is invertible.
- ▶ When $AB = I_n$ we say that B is a *right inverse* of A , and that A is a left inverse of B . A^{-1} is thus both a left and right inverse of A . $BA\vec{x} = B\vec{0} = \vec{0} \Rightarrow \vec{x} = \vec{0}$
- ▶ If $BA = I_n$, $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$, so that A satisfies 1.-4., and is thus invertible. But then $B = A^{-1}$, which also is invertible and also satisfies 1.-4.. $BA = I_n$
- ▶ Similar reasoning holds if $AB = I_n$. $BA A^{-1} = A^{-1} \Rightarrow B = A^{-1}$.
- ▶ Thus, A satisfies 1.-4. in the invertible matrix theorem if and only if there exists a matrix B so that $AB = I_n$ (i.e., A has a right inverse). B is then automatically a left inverse as well.

Theorem (Invertible matrix theorem)

The following are equivalent for an $n \times n$ -matrix A :

- 1. A is row equivalent with I_n .*
- 2. $A\mathbf{x} = \mathbf{0}$ has $\mathbf{x} = \mathbf{0}$ as the only solution.*
- 3. $A\mathbf{x} = \mathbf{b}$ has a unique solution for all right hand sides $\mathbf{b}$.*
- 4. A can be written as a product of elementary matrices.*
- 5. A is invertible.*

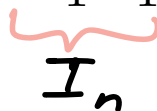
Invertibility and elementary matrices

- ▶ If A is invertible we can write it as $A = E_k \cdots E_1$, where the E_i are elementary matrices
- ▶ Write $B = F_1 \cdots F_k$, where the F_i are the reverse counterparts of the E_i .

F_i is the inverse of E_i and B is the inverse of A since

$$BA = F_1 \cdots F_k E_k \cdots E_1 = I_n,$$

$$AB = E_k \cdots E_1 F_1 \cdots F_k = I_n,$$


$$I_n$$

$$E_2 F_2 = I_n \cdots$$

Some properties of the inverse matrix

► $(A^{-1})^{-1} = A$

► if $s \neq 0$: $(sA)^{-1} = s^{-1}A^{-1}$

$$A^{-1}A = AA^{-1}$$

$$(sA)(s^{-1}A^{-1}) = ss^{-1}AA^{-1} = I_n \Rightarrow (sA)^{-1} = s^{-1}A^{-1}$$

$$(A^T)^{-1} = (A^{-1})^T : A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$$

Computing A^{-1}

Apply the row operations not only on A , but also on the augmented matrix $(A \ I_n)$.

If $E_k \cdots E_1 A = I_n$ we get $A^{-1} = E_k \cdots E_1$, and

$$E_k \cdots E_1 (A \ I_n) = (I_n \ E_k \cdots E_1) = (I_n \ A^{-1}).$$

From the result we can read out A^{-1} from the last columns.

Proposition

Suppose that A is invertible. The reduced echelon form of $(A \ I_n)$ equals $(I_n \ A^{-1})$.

Alternative interpretation: when row reducing $(A \ I_n)$, we simultaneously solve the systems

$$A\mathbf{x}_1 = \mathbf{e}_1 \quad A\mathbf{x}_2 = \mathbf{e}_2 \quad \dots\dots \quad A\mathbf{x}_n = \mathbf{e}_n,$$

and since $A\mathbf{x}_i = \mathbf{e}_i$ implies that $A^{-1}\mathbf{e}_i = \mathbf{x}_i$, column i in A^{-1} will be $\mathbf{x}_i$:

Example

Find the inverse of $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 4 & 1 \\ 2 & 4 & 2 \end{pmatrix}$.

$$(A \ I_n) = \begin{pmatrix} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 & 1 & 0 \\ 2 & 4 & 2 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 1 & 0 & 1 & 0 \\ \textcircled{2} & 3 & 1 & 1 & 0 & 0 \\ \textcircled{2} & 4 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 4 & 1 & 0 & 1 & 0 \\ 0 & -5 & -1 & 1 & -2 & 0 \\ 0 & -4 & 0 & 0 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & \textcircled{-5} & -1 & 1 & -2 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 4 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & -1 & 1 & \frac{1}{2} & -\frac{5}{4} \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & \textcircled{1} & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & -1 & -\frac{1}{2} & \frac{5}{4} \end{pmatrix}$$

$$\sim \begin{pmatrix} \textcircled{4} & 0 & 1 & \frac{3}{2} & \frac{5}{4} & \frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & -1 & -\frac{1}{2} & \frac{5}{4} \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & -1 & -\frac{1}{2} & \frac{5}{4} \end{pmatrix}$$



 A^{-1}

$$A^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{4} \\ -1 & -\frac{1}{2} & \frac{5}{4} \end{pmatrix}$$

Verification 1-2

With the built-in function for computing the inverse:

```
np.linalg.inv([[2,3,1],[1,4,1],[2,4,2]])
```

```
array([[ 1.   , -0.5   , -0.25],  
       [ 0.   ,  0.5   , -0.25],  
       [-1.   , -0.5   ,  1.25]])
```

Symbolically:

```
sym.Matrix([[2,3,1],[1,4,1],[2,4,2]]).inv()
```

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{4} \\ -1 & -\frac{1}{2} & \frac{5}{4} \end{bmatrix}$$

Verification 3

With the method `gaussian_elimination`, which we implemented:

```
matr = np.block([[np.array([[2,3,1],[1,4,1],[2,4,2]]), dtype=  
gaussian_elimination(matr, verbose=False)  
matr[:,3:]
```

```
array([[ 1.   , -0.5  , -0.25],  
       [ 0.   ,  0.5  , -0.25],  
       [-1.   , -0.5  ,  1.25]])
```

Some of the elementary matrices

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

and

$$F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

With these we have that $A^{-1} = E_9 \cdots E_1$, and $A = F_1 \cdots F_9$.

Example

Consider the system $A\mathbf{x} = \mathbf{b}$ with $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 4 & 1 \\ 2 & 4 & 2 \end{pmatrix}$ from the

previous example, and where $\mathbf{b} = \begin{pmatrix} 19 \\ 21 \\ 26 \end{pmatrix}$. We found A^{-1} above.

Solution of the system is

$$\begin{aligned} \mathbf{x} = A^{-1}\mathbf{b} &= \begin{pmatrix} 1 & -1/2 & -1/4 \\ 0 & 1/2 & -1/4 \\ -1 & -1/2 & 5/4 \end{pmatrix} \begin{pmatrix} 19 \\ 21 \\ 26 \end{pmatrix} \\ &= \begin{pmatrix} 19 - \frac{21}{2} - \frac{13}{2} \\ \frac{21}{2} - \frac{13}{2} \\ -19 - \frac{21}{2} + \frac{65}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \end{aligned}$$

Gaussian elimination on the augmented matrix $(A \quad \mathbf{b})$ is to prefer, since $(A \quad I_n)$ has more columns than $(A \quad \mathbf{b})$.

The Vandermonde matrix is invertible

We will prove by induction on n that

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \quad (1)$$

is invertible if the generators $x_0, x_1, \dots, x_n$ are distinct:

The Fourier matrix is invertible

- ▶ Follows since it is a Vandermonde matrix in particular.
- ▶ We have a general expression for its inverse, in contrast to the Vandermonde matrix:

Let B the matrix with components $b_{kn} = \frac{1}{N} e^{2\pi i kn/N}$. We compute

$$(F_N B)_{kl} =$$