

Foils for Chapter 5 in linear algebra and numerical methods

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Linear independence and bases. Basic definitions

Let $\{\mathbf{a}_i\}_{i=1}^n$ be vectors in $\mathbb{R}^m$, and $c_1, c_2, \dots, c_n$ be scalars.

- ▶ $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \dots + c_n\mathbf{a}_n$ is called a *linear combination* of $\{\mathbf{a}_i\}_{i=1}^n$.
- ▶ The *Span* of $\{\mathbf{a}_i\}_{i=1}^n$, written $\text{Sp}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)$, is the collection of all linear combinations of $\{\mathbf{a}_i\}_{i=1}^n$. We say that the $\mathbf{a}_i$ *span* $\mathbb{R}^m$ if $\text{Sp}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n) = \mathbb{R}^m$.
- ▶ We also say that they are *linearly independent* if every $\mathbf{b} \in \text{Sp}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)$ can be written as a linear combination of the $\mathbf{a}_i$ in a unique way.

Proposition

The following are equivalent:

1. $\{\mathbf{a}_i\}_{i=1}^n$ are linearly independent
2. $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \dots + c_n\mathbf{a}_n = \mathbf{0}$ always implies that $c_1 = c_2 = \dots = c_n = 0$.

A relation on the form

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \dots + c_n\mathbf{a}_n = \mathbf{0}$$

is also called a *linear dependence relation*. In other words, if $\mathbf{c} = (c_1, \dots, c_n)$, then $A\mathbf{c} = \mathbf{0}$ is a linear dependence relation between the columns of A .

Proof:

1. $\Rightarrow$ 2.

Anta $\{\vec{a}_i\}_{i=1}^n$ er lineært uafhængige, og anta

$$\text{at } c_1\vec{a}_1 + c_2\vec{a}_2 + \dots + c_n\vec{a}_n = \vec{0} \quad (1)$$

en løsning er at $c_1 = c_2 = \dots = c_n = 0$

Siden $\vec{0}$ kan skrives unikt slik
(siden $\{a_i, \vec{x}_{i=1}^n$ er lineært uavhengige),

så følger at $c_1 = c_2 = \dots = 0$ er den unike løsningen av (1),
og 2. følger

2. $\Rightarrow$ 1. : Anta kontrapositivt at 1. er usant, det
vil si at $\{a_i, \vec{x}_{i=1}^n$ er lineært avhengige.

Da finnes en vektor $\vec{b}$ som kan skrives på to forskjellige
måter:

$$\vec{b} = c_1 \vec{a}_1 + \dots + c_n \vec{a}_n$$

$$\vec{b} = d_1 \vec{a}_1 + \dots + d_n \vec{a}_n$$

Trekk disse fra hverandre:

$$\vec{0} = (c_1 - d_1) \vec{a}_1 + \dots + (c_n - d_n) \vec{a}_n$$

Sett $t_k = c_k - d_k$: $\vec{0} = t_1 \vec{a}_1 + \dots + t_n \vec{a}_n$, der ikke alle t_i er 0.

Det eksisterer altså $(c_1, \dots, c_n)$ s.d.

$$t_1 \vec{a}_1 + \dots + t_n \vec{a}_n = \vec{0}$$

Det følger at 2. $\Rightarrow$ 1. (ikke 1. $\Rightarrow$ ikke 2.)

Orthogonal sets of vectors are linearly independent.

Anta $c_1 \vec{a}_1 + \dots + c_n \vec{a}_n = \vec{0}$, der $\vec{a}_i$ er ortogonale.

Ta skalarprodukt med $\vec{a}_i$ på begge sider:

$$(c_1 \vec{a}_1 + \dots + c_n \vec{a}_n) \cdot \vec{a}_i = 0$$

$$c_1 \underbrace{(\vec{a}_1 \cdot \vec{a}_i)}_0 + \dots + c_{i-1} \underbrace{(\vec{a}_{i-1} \cdot \vec{a}_i)}_0 + c_i \underbrace{(\vec{a}_i \cdot \vec{a}_i)}_{\neq 0} + c_{i+1} \underbrace{(\vec{a}_{i+1} \cdot \vec{a}_i)}_0 + \dots + c_n \underbrace{(\vec{a}_n \cdot \vec{a}_i)}_0$$

$$c_i (\vec{a}_i \cdot \vec{a}_i) = 0$$

$$c_i |\vec{a}_i|^2 = 0$$

$$c_i = 0$$

Det følger at $c_1 = c_2 = \dots = c_n = 0$,

slik at $\{\vec{a}_i\}_{i=1}^n$ er lineært uavhengige. ■

Proposition

Row equivalent matrices share the same linear dependence relations between columns. In particular, linear dependence relations between columns in a matrix are preserved by elementary row operations.

Proof: Vi antar at $A \sim B$. Da er $B = E_k \dots E_2 E_1 A$.
Elementar.

Vi vet fra før at: $Ax = \vec{b}$ og $E A x = E \vec{b}$ har de samme løsningene.

Sett $\vec{b} = \vec{0}$. Da får vi:

$Ax = \vec{0}$ og $E A x = \vec{0}$ har de samme løsningene.

Lin. uavhengighetsrelasjon
mellom søylene i A

Lin. uavhengighetsrelasjon
mellom søylene i EA

Siden A og EA har de samme lineære uavhengighetsrelasjoner,
så har også A og B det ($B = E_k \dots E_2 E_1 A$)

Linear independence, span, and pivot elements 1.

- ▶ $\mathbf{b}$ is a linear combination of $\{\mathbf{a}_i\}_{i=1}^n$: we can find $x_1, x_2, \dots, x_n$ so that

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \dots x_n \mathbf{a}_n = \mathbf{b},$$

i.e., $A\mathbf{x} = \mathbf{b}$ has a solution, i.e., the last column in the augmented matrix $(A \ \mathbf{b})$ is not a pivot column.

- ▶ $\mathbf{b}$ can be written as a linear combination of $\{\mathbf{a}_i\}_{i=1}^n$ in a unique way: requires in addition that all columns in A are pivot columns.
- ▶ $\{\mathbf{a}_i\}_{i=1}^n$ span $\mathbb{R}^m$: $A\mathbf{x} = \mathbf{b}$ has always a solution, i.e., all rows in A are pivot rows. (this implies that the last column in $(A \ \mathbf{b})$ never becomes a pivot column).
- ▶ In a reduced echelon form: pivot columns are linearly independent. The other columns can be expressed in terms of these. This therefore also applies for the matrix itself - the pivot columns are linearly independent, and the other columns are linear combinations of these.

Linear independence, span, and pivot elements 2.

Proposition

Sett $A = (\vec{a}_1, \dots, \vec{a}_n)$. A er $m \times n$,
med $\vec{a}_i$ som søyler

The following hold

- ▶ If $\{\vec{a}_i\}_{i=1}^n$ span $\mathbb{R}^m$ then we have that $n \geq m$.
- ▶ If $\{\vec{a}_i\}_{i=1}^n$ are linearly independent we have that $n \leq m$.

Proof: Anta $\{\vec{a}_i\}_{i=1}^n$ utspenner $\mathbb{R}^m$

Da er alle rader i A pivotrader, og

$$m = \# \text{rader} = \# \text{pivotrader} = \# \text{pivot søyler} \leq \# \text{søyler} = n$$

Derfor er $m \leq n$

Anta $\{\vec{a}_i\}_{i=1}^n$ er lineært uavhengige.

Da er alle søylene i A pivotsøyler, og

$$n = \# \text{søyler} = \# \text{pivotsøyler} = \# \text{pivotrader} \leq \# \text{rader} = m$$

Derfor er $n \leq m$



Bases

A *basis* for $\mathbb{R}^m$ is a linearly independent set of vectors which span $\mathbb{R}^m$. Bases are denoted by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and so on. From the result above: a basis for $\mathbb{R}^m$ has exactly m elements. $\mathbb{R}^m$ has in particular the simple basis consisting of

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \dots, \mathbf{e}_m = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

We will denote this by $\mathcal{E}$, and call it the *standard basis*.

Proposition (Invertible matrix theorem)

The following are equivalent for an $n \times n$ -matrix A :

1. A is row equivalent with I_n .
2. $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$.
3. $A\mathbf{x} = \mathbf{b}$ has a unique solution for all right hand sides $\mathbf{b}$.
4. A can be written as a product of elementary matrices.
5. A is invertible.
6. The columns of A are a basis for $\mathbb{R}^n$.

Proof: 6. er det samme som at alle rader og søyler har pivotelement, som er det samme som 1.

Example: Find a linearly independent subset

We have the vectors

$$\mathbf{a}_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \mathbf{a}_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad \mathbf{a}_4 = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}.$$

We show that these span $\mathbb{R}^3$ by row-reducing to echelon form:

$$\begin{pmatrix} 0 & 1 & 1 & -2 \\ 2 & 1 & 2 & 3 \\ 1 & 2 & 0 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 4 \\ 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -5 \\ 0 & 1 & 1 & -2 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & -3 & 2 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 5 & -11 \end{pmatrix}$$

Trenger ikke bringe helt til (reduert) trappeform:

pivotsøyler: søyle 1, 2, 3

pivotrader: 1, 2, 3.

We observe that:

- ▶ In the echelon form we see that all rows are pivot rows, so that the four vectors span $\mathbb{R}^3$.
- ▶ The first three columns are pivot columns, so that these are linearly independent.
- ▶ The fourth vector can be expressed with the other three columns since it is not a pivot column.
- ▶ $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are both linearly independent and span $\mathbb{R}^3$ (i.e., a basis)

Bringer til redusert trappeform:

$$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -\frac{11}{5} \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & \frac{1}{5} \\ 0 & 0 & 1 & -\frac{11}{5} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & \frac{18}{5} \\ 0 & 1 & 0 & \frac{1}{5} \\ 0 & 0 & 1 & -\frac{11}{5} \end{pmatrix} \begin{matrix} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 & \vec{c}_4 \end{matrix}$$

ser at: $\vec{c}_4 = \frac{18}{5}\vec{c}_1 + \frac{1}{5}\vec{c}_2 - \frac{11}{5}\vec{c}_3$

Derfor er også: $\vec{a}_4 = \frac{18}{5}\vec{a}_1 + \frac{1}{5}\vec{a}_2 - \frac{11}{5}\vec{a}_3$.

Sjekk: $\frac{18}{5} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \frac{1}{5} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{11}{5} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} - \frac{11}{5} \\ \frac{36}{5} + \frac{1}{5} - \frac{22}{5} \\ \frac{18}{5} + \frac{2}{5} \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$

Example: Expand to a basis

We expand

$$\mathbf{a}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\mathbf{a}_2 = \begin{pmatrix} 6 \\ -1 \\ 3 \end{pmatrix}$$

to a basis for $\mathbb{R}^3$:

Skrive ned radoperasjonene!

trenger ikke gå helt til redusert trappetform

$$\begin{pmatrix} 3 & 6 \\ 2 & -1 \\ 1 & 3 \end{pmatrix} \xrightarrow{\text{I} \leftrightarrow \text{III}} \begin{pmatrix} 1 & 3 \\ 2 & -1 \\ 3 & 6 \end{pmatrix} \xrightarrow{\text{II} - 2\text{I}} \begin{pmatrix} 1 & 3 \\ 0 & -7 \\ 3 & 6 \end{pmatrix} \xrightarrow{\text{III} - 3\text{I}} \begin{pmatrix} 1 & 3 \\ 0 & -7 \\ 0 & -3 \end{pmatrix}$$
$$\xrightarrow{\text{II} / (-7)} \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 0 & -3 \end{pmatrix} \xrightarrow{\text{III} + 3\text{II}} \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

utvid med $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ til en basis for $\mathbb{R}^3$.

ta reverse radoperasjoner i motsatt rekkefølge:

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{III} - 3\text{II}} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \xrightarrow{\text{II} \cdot (-7)} \begin{pmatrix} 1 & 3 & 0 \\ 0 & -7 & 0 \\ 0 & -3 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{III} + 3\text{I}} \begin{pmatrix} 1 & 3 & 0 \\ 0 & -7 & 0 \\ 3 & 6 & 1 \end{pmatrix} \xrightarrow{\text{II} + 2\text{I}} \begin{pmatrix} 1 & 3 & 0 \\ 2 & -1 & 0 \\ 3 & 6 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{I} \leftrightarrow \text{II}} \begin{pmatrix} 2 & -1 & 0 \\ 1 & 3 & 0 \\ 3 & 6 & 1 \end{pmatrix}$$

$\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3$

Vi legger til $\vec{a}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, og får basisen

$$\left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Subspaces

A set $U \subseteq \mathbb{R}^m$ so that

1. if $\mathbf{u}_1, \mathbf{u}_2 \in U$, then also $\mathbf{u}_1 + \mathbf{u}_2 \in U$
2. if $\mathbf{u} \in U$, then also $c\mathbf{u} \in U$ for all scalars c .

Examples of subspaces:

- ▶ $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n)$: Span of the vectors $\{\mathbf{a}_i\}_{i=1}^n$.
- ▶ $\text{Col}A$: The column space of A : the span of all the columns in the matrix the pivot columns of A are a basis for $\text{Col}(A)$.
- ▶ $\text{Row}A$: The row space of A : the span of all the rows in the matrix the pivot rows of A (the rows with a leading one) are a basis for $\text{Row}(A)$,
- ▶ $\text{Nul}A$: The null space of A : the collection of all vectors $\mathbf{x}$ so that $A\mathbf{x} = \mathbf{0}$
- ▶ $A^\perp$: The orthogonal complement of the set A .

$$A^\perp = \{ \vec{b} : \vec{b} \cdot \vec{a} = 0 \quad \forall \vec{a} \in A \}$$

For å vise at $A^\perp$ er underrom:

Må vise at $\vec{u}_1 + \vec{u}_2 \in A^\perp$ når $\vec{u}_1$ og $\vec{u}_2$ er i $A^\perp$

$$(\vec{u}_1 + \vec{u}_2) \cdot \vec{a} = \vec{u}_1 \cdot \vec{a} + \vec{u}_2 \cdot \vec{a} = 0 + 0 = 0$$

derfor er også $\vec{u}_1 + \vec{u}_2 \in A^\perp$.

$$(c\vec{u}) \cdot \vec{a} = c(\vec{u} \cdot \vec{a}) = c \cdot 0 = 0$$

derfor er også $c\vec{u} \in A^\perp$

Merk: $\text{Nul}(A) = (\text{Row } A)^\perp$

$$\left(\begin{array}{c} \text{---} \end{array} \right) \left(\begin{array}{c} \text{---} \end{array} \right)$$

- ▶ A *basis* for a subspace: a linearly independent set of vectors which spans the whole subspace.
- ▶ Any subspace of $\mathbb{R}^m$ has a basis.

From the reduced echelon form of A it follows that

- ▶ the pivot rows of A (the rows with a leading one) are a basis for $\text{Row}(A)$,
- ▶ the pivot columns of A are a basis for $\text{Col}(A)$.

Since bases of $\mathbb{R}^m$ have exactly m vectors, a subspace of $\mathbb{R}^m$ can't have bases with more than m vectors.

Example

We find bases for the row space, column space, and null space of

$$A = \begin{pmatrix} 2 & 1 & 3 & 8 \\ 1 & 3 & -1 & 4 \\ 4 & -3 & 11 & 16 \end{pmatrix}.$$

$$\begin{pmatrix} 2 & 1 & 3 & 8 \\ 1 & 3 & -1 & 4 \\ 4 & -3 & 11 & 16 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -1 & 4 \\ 2 & 1 & 3 & 8 \\ 4 & -3 & 11 & 16 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -1 & 4 \\ 0 & -5 & 5 & 0 \\ 0 & -15 & 15 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 3 & -1 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Første to rader er pivotrader $\Rightarrow \{(1 \ 0 \ 2 \ 4), (0 \ 1 \ -1 \ 0)\}$
er en basis for Row A

Første to søyler er pivotsøyler $\Rightarrow \left\{ \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} \right\}$
er en basis for $\text{Col } A$.

En generell vektor $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ i nullrommet må tilfredsstille:

$$\begin{array}{rcl} x_1 + 2x_3 + 4x_4 & = & 0 \\ x_2 - x_3 & = & 0 \end{array} \Rightarrow \begin{array}{l} x_1 = -2x_3 - 4x_4 \\ x_2 = x_3 \end{array}$$

Dette gir:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_3 - 4x_4 \\ x_3 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ er en basis for } \text{Nul } A.$$

In this example there were

- ▶ two free variables,
- ▶ a null space of dimension two.

More generally it is the case that the number of free variables always equals the dimension of the null space.

All bases for a subspace have the same number of vectors.

- ▶ The number of vectors in a basis is called the *dimension* of the subspace.
- ▶ When a matrix is on reduced echelon form we see that the dimension of both the row space and the column space equals the number of pivot elements.
- ▶ This common dimension of $\text{Col}(A)$ and $\text{Row}(A)$ is called the *rank* of the matrix, written $\text{rank}(A)$.

Proof: Antag $S_p(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n) = S_p(\vec{b}_1, \vec{b}_2, \dots, \vec{b}_k)$
der $n > k$

Se på afbildningen $\vec{F}: \mathbb{R}^n \rightarrow \mathbb{R}^k$ defineret ved

$$\vec{F}(c_1, c_2, \dots, c_n) = (d_1, d_2, \dots, d_k)$$

$$\text{når } c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_n \vec{a}_n = d_1 \vec{b}_1 + d_2 \vec{b}_2 + \dots + d_k \vec{b}_k$$

Denne er veldefineret siden $\sum_{i=1}^n \vec{a}_i$ er lineært uafhængige,
og $\sum_{i=1}^k \vec{b}_i$ er lineært uafhængige.
 $\vec{F}$ er defineret på hele $\mathbb{R}^n$.

$\vec{F}$ er også injektiv.

$\vec{F}$ er linear (vanskeligere):

$$(c_1, \dots, c_n) \xrightarrow{\vec{F}} (d_1, \dots, d_k) \Rightarrow c_1 \vec{a}_1 + \dots + c_n \vec{a}_n = d_1 \vec{b}_1 + \dots + d_k \vec{b}_k$$

$$(e_1, \dots, e_n) \rightarrow (f_1, \dots, f_k) \Rightarrow e_1 \vec{a}_1 + \dots + e_n \vec{a}_n = f_1 \vec{b}_1 + \dots + f_k \vec{b}_k.$$

legg sammen:

$$(c_1 + e_1) \vec{a}_1 + \dots + (c_n + e_n) \vec{a}_n = (d_1 + f_1) \vec{b}_1 + \dots + (d_k + f_k) \vec{b}_k$$

$$\vec{F}(c_1 + e_1, \dots, c_n + e_n) = (d_1 + f_1, \dots, d_k + f_k)$$

$$\vec{F}(\vec{c} + \vec{e}) = \vec{d} + \vec{f}$$

Vi har derfor vist første del av linearitet.

$\vec{F}$ kan derfor uttrykkes med en $k \times n$ -matrise A ,
slik at $\vec{F}(\vec{x}) = A\vec{x}$. (A er $k \times n$)

Siden $n > k$ har A en søyle som ikke er en pivotsøyle.

$\Rightarrow$ har en uavhengighetsrelasjon mellom søylene i A
 $x_1 \vec{a}_1 + \dots + x_n \vec{a}_n = \vec{0}$. Men da er $A\vec{x} = \vec{0}$, der $\vec{x} \neq \vec{0}$

Example: Rank one matrices

$$\vec{F}(\vec{x}) = \vec{0}$$

Siden også $\vec{F}(\vec{0}) = \vec{0}$, betyder dette at $\vec{F}$ er injektiv.

Det følger at $n = k$, slik at det er like mange vektorer i de to basisene.

Eksempel: Rang 1-matriser:

La $\vec{a} \in \mathbb{R}^m$, $\vec{b} \in \mathbb{R}^n$. Da er

$$\vec{a} \vec{b}^T = \begin{pmatrix} a_1 b_1 & \dots & a_1 b_n \\ \vdots & & \vdots \\ a_m b_1 & \dots & a_m b_n \end{pmatrix} = \begin{pmatrix} b_1 \vec{a} & \dots & b_n \vec{a} \end{pmatrix} = \begin{pmatrix} a_1 \vec{b}^T \\ \vdots \\ a_m \vec{b}^T \end{pmatrix}$$

$\{\vec{a}\}$ er basis for $\text{Col } A$

$\{\vec{b}^T\}$ er basis for $\text{Row}(A)$

$\vec{a} \vec{b}^T$ har derfor rang 1.

Alle rang 1-matriser kan skrives på formen $\vec{a} \vec{b}^T$

: La $\{\vec{a}\}$ være en basis for $\text{Col}(A)$. Da kan vi skrive:

$$A = (\vec{a}_1 \dots \vec{a}_r) \begin{pmatrix} \vec{b}_1^T & \dots & \vec{b}_r^T \end{pmatrix}$$

$$A = (b_1 \vec{a} \cdots b_n \vec{a}) = \vec{a} b,$$

Proposition

Let A be an $m \times n$ -matrix. Then we have that

$$\text{rank}(A) + \dim(\text{Nul}(A)) = n.$$

In the previous example: the rank was 1, and the dimension of the null space was $n - 1$. This agrees with the proposition, since $1 + (n - 1) = n$.

Proof:

$$n = \# \text{spøyles} = \underbrace{\# \text{pivotspøyles}}_{\text{rank } A} + \underbrace{\# \text{frie variable}}_{\dim(\text{Nul } A)}.$$

$$\text{Derfor: } n = \text{rank } A + \dim(\text{Nul } A).$$

The reduced echelon form is unique.

Proof:

Anta for motsigelse at C_1 og C_2 er to forskjellige reduserte trappetformer for matrisen A

Vi har at $C_1 \sim C_2 \Rightarrow C_1$ og C_2 har de samme lineære uavhengighetsrelasjonene mellom søylene sine.

1. Viser først: C_1 og C_2 har de samme posisjonene for pivotsøyer.

Hvis ikke: Finnes en første indeks k slik at søyle k er pivotsøyle i C_1 , men ikke i C_2

↓
har lineær uavhengighetsrelasjon mellom disse søylene i C_2 .

har ingen slik uavhengighetsrelasjon i C_1

Motsigelse, siden $C_1 \sim C_2$

2. De resterende søylene kan skrives som lineære kombinasjoner av pivotsøylene.

Siden C_1 og C_2 har de samme lineære avhengighetsrelasjonene, og har de samme pivotsøylene, så er også disse søylene like.

Ekamen 2025 (avsluttende)

Oppgave 4

$$a) A = \begin{pmatrix} 4 & 2 & 4 & 14 \\ 1 & -1 & 7 & -1 \\ 3 & 3 & -3 & 15 \end{pmatrix} \xrightarrow{I \leftrightarrow II} \begin{pmatrix} 1 & -1 & 7 & -1 \\ 4 & 2 & 4 & 14 \\ 3 & 3 & -3 & 15 \end{pmatrix}$$

$$\xrightarrow{II - 4I} \begin{pmatrix} 1 & -1 & 7 & -1 \\ 0 & 6 & -24 & 18 \\ 3 & 3 & -3 & 15 \end{pmatrix} \xrightarrow{III - 3I} \begin{pmatrix} 1 & -1 & 7 & -1 \\ 0 & 6 & -24 & 18 \\ 0 & 6 & -24 & 18 \end{pmatrix}$$

$$\xrightarrow{III - II} \begin{pmatrix} 1 & -1 & 7 & -1 \\ 0 & 6 & -24 & 18 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

ser at søyle 1 og 2 blir pivotsøyle,

slik at $\left\{ \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right\}$ er en basis for $\text{col}(A)$.

b) Setter inn en siste søyle:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{III + II} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 6 & 0 \\ 0 & 6 & 1 \end{pmatrix} \xrightarrow{III - II} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{II + 4I} \begin{pmatrix} 1 & -1 & 0 \\ 4 & 2 & 0 \\ 3 & 3 & 1 \end{pmatrix} \xrightarrow{I \leftrightarrow II} \begin{pmatrix} 4 & 2 & 0 \\ 1 & -1 & 0 \\ 3 & 3 & 1 \end{pmatrix}$$

ny søyle

kan utvide med $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ til basisen $\left\{ \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

konteeksamen 2025 (avsluttende)

Oppg. 4: a) Nullrommet til $A = \begin{pmatrix} 2 & 1 & 8 & 7 \\ 1 & 0 & 3 & 4 \\ 4 & 3 & 18 & 13 \end{pmatrix}$?

$$\begin{pmatrix} 2 & 1 & 8 & 7 \\ 1 & 0 & 3 & 4 \\ 4 & 3 & 18 & 13 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & 4 \\ 2 & 1 & 8 & 7 \\ 4 & 3 & 18 & 13 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 3 & 6 & -3 \end{pmatrix} \\ \sim \begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

nullrom ($A\vec{x} = \vec{0}$):

$$\left. \begin{aligned} x_1 + 3x_3 + 4x_4 &= 0 \\ x_2 + 2x_3 - x_4 &= 0 \end{aligned} \right\} \begin{aligned} x_1 &= -3x_3 - 4x_4 \\ x_2 &= -2x_3 + x_4 \end{aligned}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3x_3 - 4x_4 \\ -2x_3 + x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$\left\{ \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ er en basis for nullrommet

b) Utvid til basis for $\mathbb{R}^4$:

$$\begin{pmatrix} -3 & -4 \\ -2 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{I \leftrightarrow III} \begin{pmatrix} 1 & 0 \\ -2 & 1 \\ -3 & -4 \\ 0 & 1 \end{pmatrix} \xrightarrow{\begin{aligned} II + 2I \\ III + 3I \end{aligned}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -4 \\ 0 & 1 \end{pmatrix} \xrightarrow{\begin{aligned} III + 4II \\ IV - II \end{aligned}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\begin{aligned} III - 4II \\ IV + II \end{aligned}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{\begin{aligned} II - 2I \\ III - 3I \end{aligned}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & -4 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{I \leftrightarrow III} \begin{pmatrix} -3 & -4 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

nye spylar.

V_i kan utvise basisen til

$$\left\{ \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$
