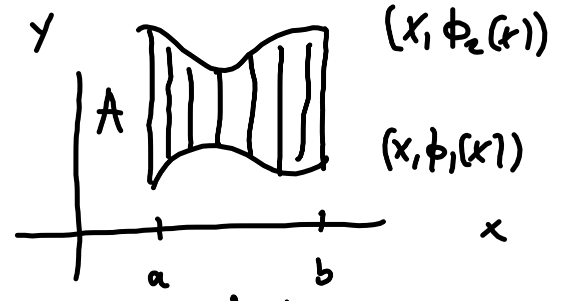
6.2

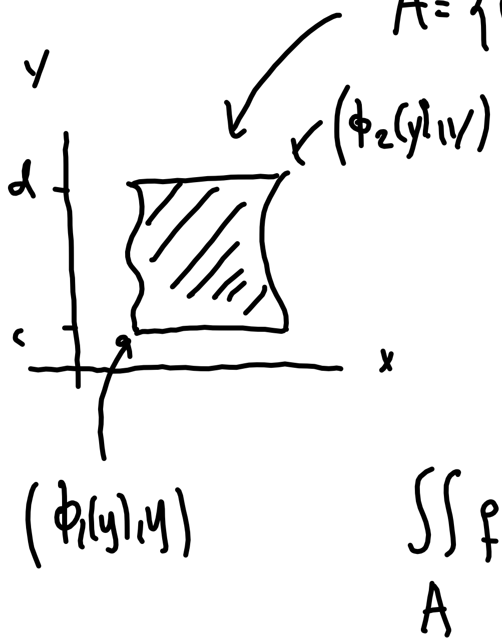
Type 1-område:



$$\iint_A f(x,y)\,dx\,dy = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right) dx.$$

Type 2-område: La $[c,d]$ være et intervall. La $\phi_1, \phi_2 : [c,d] \to \mathbb{R}$ være kontinuerlige funksjoner s.a. $\phi_1(y) \le \phi_2(y)$.

$$A = \{ (x,y) \in \mathbb{R}^2 : y \in [c,d],\ \phi_1(y) \le x \le \phi_2(y) \}.$$

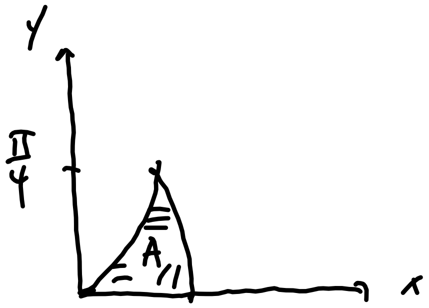


La f være en kontinuerlig funksjon på A.

Da er

$$\iint_A f(x,y)\,dx\,dy = \int_c^d \left(\int_{\phi_1(y)}^{\phi_2(y)} f(x,y)\,dx \right) dy.$$

Eks: La $[c, d] = [0, \frac{\pi}{4}]$



$\phi_1(y) = \sin y$

$\phi_2(y) = \cos y$

$f(x,y) = x \cdot y.$

$$\iint_A f(x,y)\,dx\,dy = \int_0^{\frac{\pi}{4}} \left(\int_{\sin x}^{\cos x} xy\,dx \right) dy$$

$$= \frac{1}{2} \int_0^{\pi/4} \left[x^2 y \right]_{\sin y}^{\cos y} da$$

$$= \frac{1}{2} \int_0^{\pi/4} y(\cos^2 y - \sin^2 y)\,dy$$

$$= \frac{1}{2} \int_0^{\pi/4} y \cdot \cos 2y\,dy$$

$$\begin{pmatrix} u = y & v' = \cos 2y \\ u' = 1 & v = \frac{1}{2}\sin 2y \end{pmatrix}$$

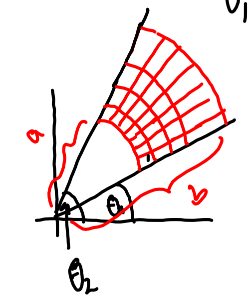
$$= \frac{1}{2} \left[\left[\frac{y}{2} \sin 2y \right]_0^{\pi/4} - \frac{1}{2} \int_0^{\pi/4} \sin 2y \right]$$

$= \ldots\ldots$

Integrasjon i polarkoordinater

La A være området som i polar-koordinater er beskrevet ved

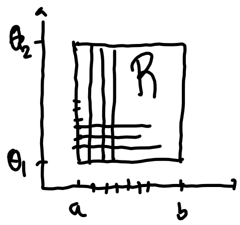
$$a \le r \le b$$
$$\theta_1 \le t \le \theta_2 .$$



La f være en kontinuerlig funksjon på A.

Lag partisjon:

$$a = r_0 < r_1 < \dots < r_n = b$$
$$\theta_1 = t_0 < t_1 < \dots < t_m = \theta_2 .$$



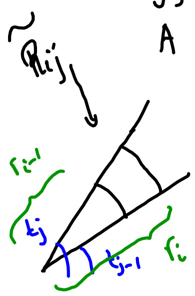
$$R_{ij} = [r_{i-1}, r_i] \times [t_{j-1}, t_j]$$

$\tilde{R}_{ij}$ tilsvarende "rektangel" oppe

Velg $c_{ij} \in \tilde{R}_{ij}$ for $1 \le i \le n, 1 \le j \le m$,

$$\iint_A f(x,y)\,dx\,dy \simeq \sum_{i,j} f(c_{ij}) \cdot \text{Areal}(\tilde{R}_{ij}).$$

dersom maskevidden er liten



$$\text{Areal}(\tilde{R}_{ij}) = \frac{1}{2}(t_j - t_{j-1})(r_i^2 - r_{i-1}^2)$$

$$= \frac{1}{2}(t_j - t_{j-1})(r_i + r_{i-1})(r_i - r_{i-1})$$

$$\simeq r_{i-1} \cdot (t_j - t_{j-1}) \cdot (r_i - r_{i-1})$$

Så $$\iint_A f(x,y)\,dx\,dy \simeq \sum_{i,j} f(c_{ij}) \cdot r_{i-1} (t_j - t_{j-1})(r_i - r_{i-1})$$

$\simeq$ Riemann-sum for

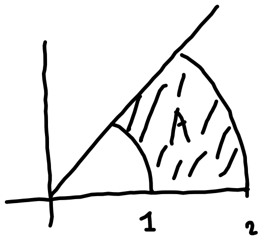
$$\iint_R f(r\cos(t), r\sin(t)) \cdot r \, dr\,dt$$

SETNING: La $0 \le \theta_1 < \theta_2 \le 2\pi$, og
$0 \le a < b < \infty$,
og la A være området som i polarkoordinater er beskrevet ved $\theta_1 \le t \le \theta_2$, $a \le r \le b$.
La f være en kontinuerlig funksjon på A. Da er

$$\iint_A f(x,y)\,dx\,dy = \iint_R f(r\cos t, r\sin t)\cdot r\,dr\,dt,$$

der $R = [a,b]\times[\theta_1,\theta_2]$.

Eks 1: $0 \le t \le \pi/4$, $1 \le r \le 2$, $f(x,y) = x^2 \cdot y$.



$$\iint_R f(x,y)\,dx\,dy$$

$$= \int_0^{\pi/4}\int_1^2 r^2\cos^2 t \cdot r\sin t \cdot r\,dr\,dt$$

$$= \int_0^{\pi/4}\int_1^2 r^4\cos^2 t\cdot \sin t\,dr\,dt$$

$$= \left(\frac{2^5-1}{5}\right)\int_0^{\pi/4}\cos^2 t\cdot\sin t\,dt.$$

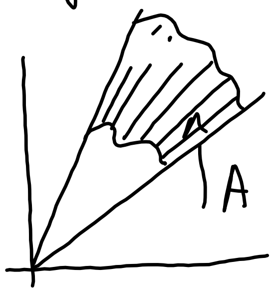
$$\int_1^2 r^4\,dr = \left[\frac{r^5}{5}\right]_1^2$$

$$= \left(\frac{2^5-1}{5}\right)\cdot\left[-\frac{1}{3}\cos^3 t\right]_0^{\pi/4} =$$

$$= \left(\frac{2^5-1}{5}\right)\cdot\left(-\frac{1}{3}\right)\cdot\left[\left(\frac{\sqrt{2}}{2}\right)^3 - 1\right].$$

Mer generelt: $\theta_1 \le t \le \theta_2$

$\varphi_1(t) \le r \le \varphi_2(t)$.



$\varphi_1, \varphi_2 : [\theta_1, \theta_2] \to \mathbb{R}$

kontinuerlige funksjoner

med $\varphi_1(t) \le \varphi_2(t)$.

For f kontinuerlig på A:

$$\iint_A f(x,y)\,dx\,dy = \int_{\theta_1}^{\theta_2}\left(\int_{\varphi_1(t)}^{\varphi_2(t)} f(r\cdot\cos t, r\cdot\sin t)\cdot r\,dr\right)dt.$$

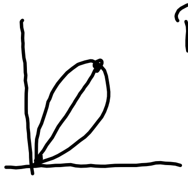
Eks: La A være området begrenset av den parametriserte kurven

$$r(t) = ((\cos t\cdot\sin t)\cdot\cos t, (\cos t\cdot \sin t)\cdot\sin t)$$

$$= \cos t\cdot\sin t\cdot(\cos t, \sin t).$$

for $t \in [0, \frac{\pi}{2}]$.

Regn ut arealet til A.



$$\iint_A 1\cdot dx\,dy = \text{arealet}.$$

$$= \int_0^{\pi/2}\left(\int_0^{\cos t\sin t} r\,dr\right)dt$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[r^2\right]_0^{\cos t\sin t} dt = \frac{1}{2}\int_0^{\pi/2} \underbrace{\cos^2 t}_{=1-\sin^2 t}\cdot\sin^2 t\,dt.$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^2 t - \sin^4 t\,dt.$$

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Først $\int_0^{\pi/2} \sin^2 t \, dt$

$$= \frac{1}{2}\int_0^{\pi/2} 1 - \cos(2t) \, dt = \cdots$$

$$\sin^2 t = \frac{1-\cos(2t)}{2}$$

Så $\int_0^{\pi/2} \sin^4 t \, dt$

$$\left(\begin{array}{ll} u = \sin^3 t & v' = \sin t \\ u' = 3\sin^2 t \cdot \cos t & v = -\cos t \end{array} \right)$$

$$= -\underbrace{\left[\sin^3 t \cdot \cos t\right]_0^{\pi/2}}_{=0} + 3\int_0^{\pi/2} \sin^2 t \underbrace{\cos^2 t}_{=1-\sin^2 t} \, dt .$$

$$\int_0^{\pi/2} \sin^4 t \, dt = 3 \cdot \int_0^{\pi/2} \sin^2 t - 3\int_0^{\pi/2} \sin^4 t \, dt .$$

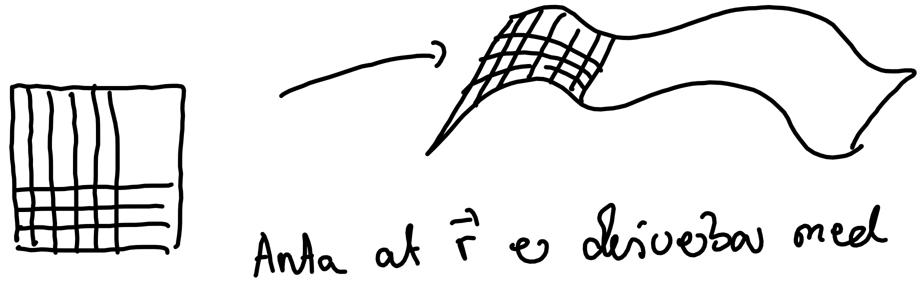
$$\int_0^{\pi/2} \sin^4 t = \frac{3}{4}\int_0^{\pi/2} \sin^2 t \, dt$$

6.4 Anvendelser.

Arealer til parametriserte flater.

$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$.

$\mathbb{R}^3$



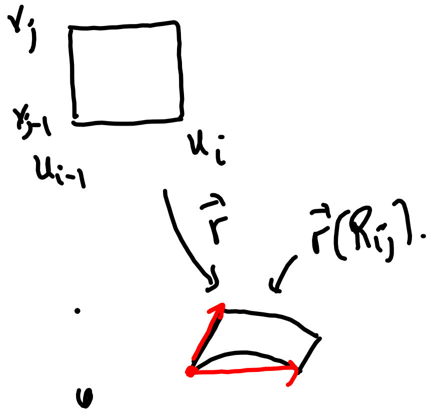
R

Anta at $\vec{r}$ er deriverbar med kontinuerlige partiellderiverte.

Lag en partisjon av R.

$$\text{Areal}\left(\vec{r}(R)\right) = \sum_{i,j} \text{Areal}\left(\vec{r}(R_{ij})\right)$$

skal estimere disse.



- Arealet til $\vec{r}(R_{ij})$ burde være omtrent det samme som arealet til parallellogrammet utspent av vektorene

$$\vec{r}(u_i, v_{j-1}) - \vec{r}(u_{i-1}, v_{j-1}) \quad \text{og} \quad \vec{r}(u_{i-1}, v_j) - \vec{r}(u_{i-1}, v_{j-1})$$

Areale til paralellogram:

$$\sum_{i,j} \frac{\left| \left(\vec{r}(u_i, v_{j-1}) + \vec{r}(u_{i-1}, v_{j-1}) \right) \times \left(\vec{r}(u_{i-1}, v_j) - \vec{r}(u_{i-1}, v_{j-1}) \right) \right|}{(u_i - u_{i-1}) \cdot (v_j - v_{j-1})} (u_i - u_{i-1})(v_j - v_{j-1})$$

$$\simeq \sum_{i,j} \left| \frac{\partial r}{\partial u}(u_{i-1}, v_{j-1}) \times \frac{\partial r}{\partial v}(u_{i-1}, v_{j-1}) \right| \cdot (u_i - u_{i-1}) \cdot (v_j - v_{j-1})$$

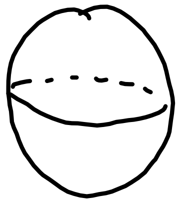
= Riemann-sum for integralet

$$\iint_R \left| \frac{\partial r}{\partial u}(u,v) \times \frac{\partial r}{\partial v}(u,v) \right| du\, dv .$$

SETNING: La $\vec{r} : [a,b] \times [c,d] \to \mathbb{R}^3$ parametrisere en flate og anta at $\vec{r}$ har kontinuerlige partiellderiverte. Da er arealet til flaten lik

$$\iint_{R=[a,b]\times[c,d]} \left| \frac{\partial r}{\partial u}(u,v) \times \frac{\partial r}{\partial v}(u,v) \right| du\,dv.$$

Eks: Regn ut arealet til overflaten av en kule med radius 1.



Fra kulekoordinater:

$$r(\phi, \theta) = (\sin\phi \cdot \cos\theta, \sin\phi \cdot \sin\theta, \cos\phi),$$

$\phi \in [0,\pi]$ og $\theta \in [0, 2\pi]$.

$$\text{Arealet} = \int_0^{\pi}\int_0^{2\pi} \left| \frac{\partial r}{\partial \phi}(\phi,\theta) \times \frac{\partial r}{\partial \theta}(\phi,\theta) \right| d\phi\, d\theta$$

$$= \cdots\cdots = \int_0^{\pi}\int_0^{2\pi} \sin\phi \; d\phi\, d\theta = 4\pi$$

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