Elektronisk orakel: orakel@math.uio.no

EKSEMPLER KJERNEREGEL.

(i) $\mathbb{R} \xrightarrow{g(t)} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}.$

$$g(t) = (\cos(t), \sin(t), t)$$

$$f(x,y,z) = x \cdot y \cdot z.$$

Finn den deriverte til $f(g(t))$ ved hjelp av kjerneregel.

$$\frac{d}{dt} f(g(t)) = f'(g(t)) \cdot g'(t)$$

- $f'(x,y,z) = \left(\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z) \right)$
  $= (yz, x \cdot z, xy)$

- $f'(g(t)) = (\sin(t) \cdot t, \cos(t) \cdot t, \cos t \cdot \sin t).$

- $g'(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{pmatrix}$

Så $\frac{d}{dt} f(g(t)) = -\sin^2 t \cdot t + \cos^2 t \cdot t + \cos(t) \cdot \sin(t).$

(Sjekk $(\sin t \cdot \cos t \cdot t)'$)

(ii) La $f(x,y) = x^2 + y^2 - R^2$, $R > 0$.

Anta at $y = g(x)$ e en funksjon s.a. $f(x, g(x)) = 0$.

Vis at $g'(x) = -\frac{x}{g(x)}$.

Inn før $h(x) = (x, g(x))$.

Da er $f(h(x)) = 0$.

- $f'(x,y) = (2x, 2y)$
- $h'(x) = \begin{pmatrix} 1 \\ g'(x) \end{pmatrix}$
- $\frac{d}{dx} f(h(x)) = f'(h(x)) \cdot h'(x) = 0$

$$= (2x, 2g(x)) \cdot \begin{pmatrix} 1 \\ g'(x) \end{pmatrix}$$

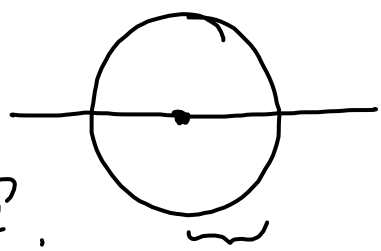
$$= 2x + 2g(x) \cdot g'(x).$$

Så $\cancel{2} \cdot g(x) \cdot g'(x) = -\cancel{2}x$

$$\underline{\underline{g'(x) = -\frac{x}{g(x)}}},$$

$f(x,y) = x^2 + y^2 - R^2 = 0$

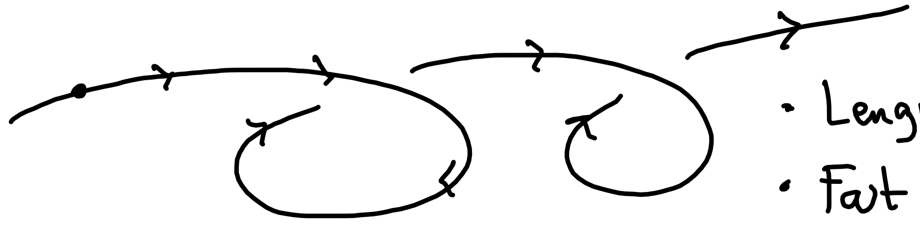
$x^2 + y^2 = R^2$



$(x, g(x))$ $\quad g(x) = \pm\sqrt{R^2 - x^2}$.

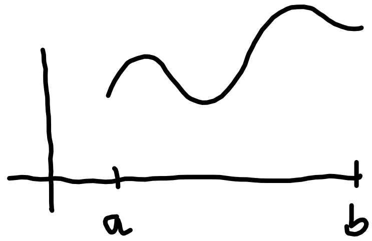
R.

Parametriserte kurver



- Lengde
- Fart
- Hastighet
- Akselerasjon.

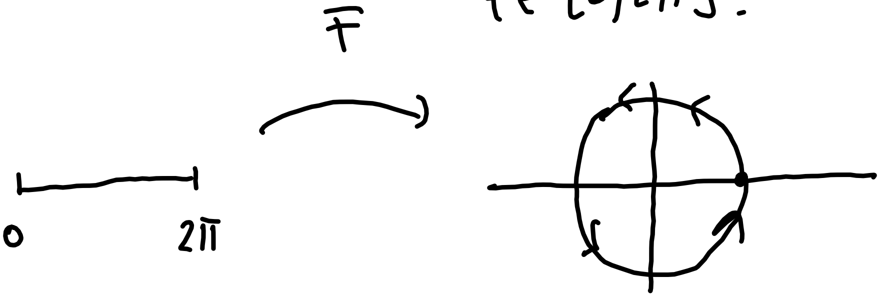
Eks 1.



$y = f(x)$,

Eksempel på en parametrisert kurve over $I = [a,b]$.

Eks 2: Definere $F(t) = (\cos(t), \sin(t))$, $t \in [0, 2\pi]$.



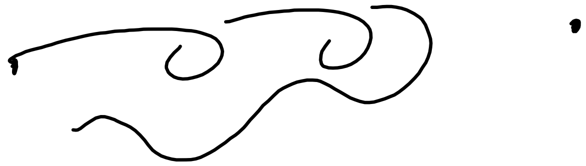
DEF 3.3.1: En <u>parametrisert kurve</u> e en kontinuerlig avbilding $\vec{r} : I \to \mathbb{R}^n$ (I e et intervall),

$r(t) = (x_1(t), x_2(t), \dots, x_n(t))$.

Eks: En bedrift fører n forskjellige varer. Lagerbeholdning

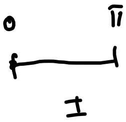
$$r(t) = (x_1(t), x_2(t), \dots, x_n(t)).$$

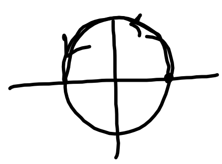
Konstant siden r er kont.



$(\cos(t), \sin(t))$

$t \in [0, \pi]$.

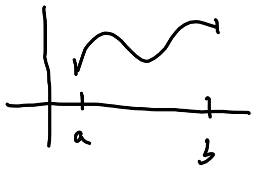




$y = \sqrt{1-x^2}$

$t \mapsto (t, \sqrt{1-t^2})$

Lengde



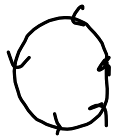
$y = g(x)$.

$$l = \int_a^b \sqrt{1+g'(x)^2}\, dx.$$

DEF: Anta at $\vec{r}(t) = (x_1(t), \dots, x_n(t))$ er en parametrisert kurve, og anta at $x_1(t), \dots, x_n(t)$ er deriverbare. Da definerer vi lengden til kurven

$$L(a,b) = \int_a^b \sqrt{x_1'(t)^2 + x_2'(t)^2 + \dots + x_n'(t)^2}\, dt.$$

Eks: La $\vec{r}(t) = (\cos(t), \sin(t))$ $t \in [0, 2\pi]$



$$L(0, 2\pi) = \int_0^{2\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2}\, dt$$

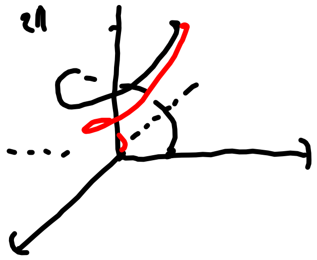
$$= \int_0^{2\pi} dt = 2\pi.$$

jan 24-10:48

Eks: La $\vec{r}(t)$ være den parametriserte kurven $(\cos(t), \sin(t), t)$

$$\vec{r}(t) = (t\cdot\cos t,\ t\sin t,\ t) \quad , \quad t \in [0, 2\pi]$$

Finn et uttrykk for lengden.



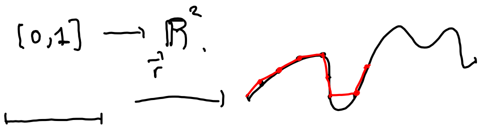
$$L(0,2\pi) = \int_0^{2\pi} \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1}\ dt$$

$$= \int_0^{2\pi} \sqrt{\cos^2 t - \cancel{2t\cos t\sin t} + t^2 \cancel{\sin^2 t} + \sin^2 t + \cancel{2t\sin t\cos t} + t^2\cancel{\cos^2 t} + 1}\ dt$$

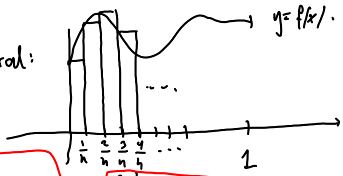
$$= \int_0^{2\pi} \sqrt{1 + t^2 + 1}\ dt = \underline{\int_0^{2\pi} \sqrt{2 + t^2}\ dt}\ .$$

Hvorfor er definisjonen av lengde rimelig?

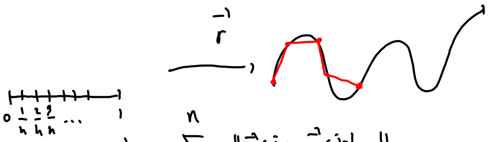
$\vec{r} : [0,1] \longrightarrow \mathbb{R}^2$.



Minne om integral:



$$\boxed{\int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{j=0}^{n-1} \frac{1}{n} f\left(\frac{j}{n}\right)}$$



$$\lim_{n\to\infty} \sum_{j=1}^{n} \left\| \vec{r}\left(\frac{j}{n}\right) - \vec{r}\left(\frac{j-1}{n}\right) \right\|$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{n} \cdot \left\| \frac{\vec{r}\left(\frac{j}{n}\right) - \vec{r}\left(\frac{j-1}{n}\right)}{1/n} \right\|$$

for stor n

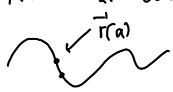
$$\approx \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{n} \left\| \vec{r}'\left(\frac{j-1}{n}\right) \right\|$$

$$= \int_0^1 \|\vec{r}'(t)\|\,dt$$

$$= \int_0^1 \sqrt{x_1'(t)^2 + \dots + x_n'(t)^2}\,dt \, .$$

Dersom $\vec{x} \in \mathbb{R}^n$ så har vi $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

Fart : $\frac{\text{lengde}}{\text{tid}}$



$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_a^{a+\varepsilon} \|\vec{r}'(t)\|\,dt$$

$$= \|\vec{r}'(a)\| .$$

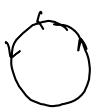
Definerer fart ved $v(t) = \|\vec{r}'(t)\|$.

Eks : $\vec{r}(t) = (\cos(t), \sin(t))$, $t \in [0, 2\pi]$.

Finn farten .

$\|\vec{r}'(t)\| = \|(-\sin(t), \cos(t))\|$

$= \sqrt{\sin^2 t + \cos^2 t} = 1$.



DEF : Hastighet er definert som $\vec{v}(t) = \vec{r}'(t)$.

DEF : Akselerasjon er definert som $\vec{a}(t) = \vec{v}'(t)$.