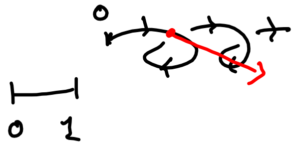
$\vec{r} : [a,b] \longrightarrow \mathbb{R}^n$



$\vec{r}(t) = (x_1(t), x_2(t), \dots, x_n(t))$

$\vec{v}(t) = \vec{r}\,'(t) = (x_1'(t), \dots, x_n'(t))$

$v(t) = \|\vec{v}(t)\|.$

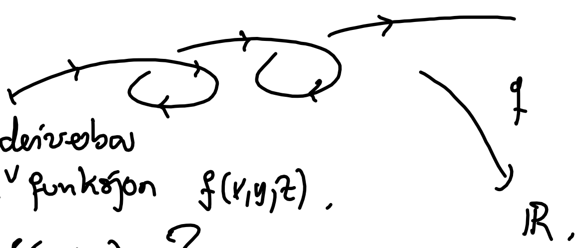
$\vec{a}(t) = \vec{v}\,'(t) = (x_1''(t), \dots, x_n''(t)).$

Oppgave: $r(t) = (\cos(t), \sin(t))$, $t \in [0, 2\pi]$.

Regn ut $\vec{v}, v, \vec{a}$.

Kjerneregelen for parametrisete kurver.

$\vec{r} : [0,1] \to \mathbb{R}^3$



La f være en deriverbar funksjon $f(x,y,z)$.

$\frac{d}{dt} f(r(t))$?

$\parallel$

$$f'(r(t)) \cdot r'(t) = \left(\frac{\partial f}{\partial x}(r(t)), \frac{\partial f}{\partial y}(r(t)), \frac{\partial f}{\partial z}(r(t)) \right) \cdot \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix}.$$

$$= \frac{\partial f}{\partial x}(r(t)) \cdot x'(t) + \frac{\partial f}{\partial y}(r(t)) \cdot y'(t) + \frac{\partial f}{\partial z}(r(t)) \cdot z'(t)$$

$$= \nabla f(r(t)) \cdot r'(t).$$

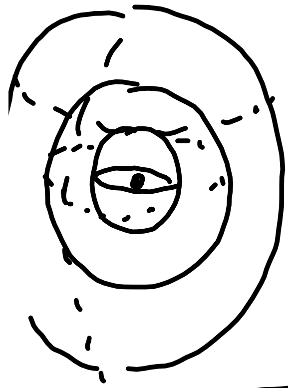
SETNING 3.2.1.: Dersom $\vec{r} : [a,b] \to \mathbb{R}^n$ er en deriverbar parametrisert kurve og dersom $f : \mathbb{R}^n \to \mathbb{R}$ er en deriverbar funksjon, så er $f \circ \vec{r}$ deriverbar, og

$$\frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot r'(t).$$

jan 27-10:07

Eksempel: Vi la temperaturen i et rom være gitt ved $T(\vec{x}) = f(\|x\|)$, også lar vi $\vec{r}(t)$ være en parametrisert kurve som beskriver en partikkel som beveger seg i rommet. Finn et uttrykk for hvordan temp. forandrer seg for partikkelen m.h.p. t.

$\|(x,y,z)\| = \sqrt{x^2+y^2+z^2}$



$$\frac{d}{dt} T(r(t)) = \underbrace{\nabla T}(r(t)) \cdot r'(t).$$

$T(\vec{x})$

↑

plugger in $\vec{r}(t)$

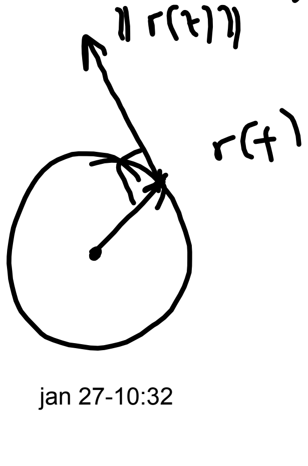
$$= \nabla f(|(x,y,z)|) = \nabla f\left(\sqrt{x^2+y^2+z^2}\right)$$

$$= f'\left(\sqrt{x^2+y^2+z^2}\right) \cdot \left(\frac{\frac{1}{2}2x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{f'(\|\vec{x}\|)}{\|\vec{x}\|}\, \vec{x}$$

$\vec{x} = (x,y,z)$

Så $\displaystyle \frac{d}{dt} T(\vec{r}(t)) = \frac{f'(\|r(t)\|)}{\|r(t)\|} \cdot r(t) \cdot r'(t).$



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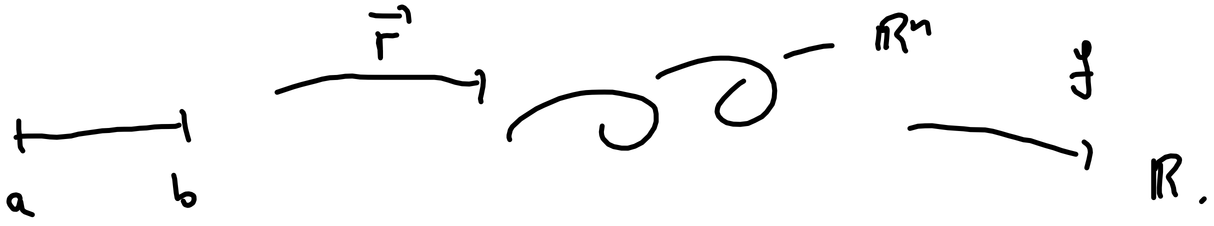


$$\vec{r}(t) = (x_1(t), x_2(t), \dots, x_n(t))$$

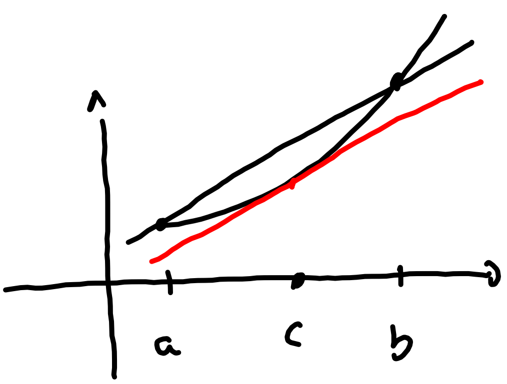
$$f(x_1, x_2, \dots, x_n) = f(\vec{x}).$$

MIDDELVERDISETNINGEN

Husk fra en variabel: f e deriverbar på $[a,b]$.

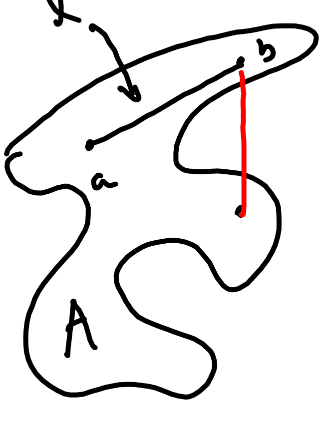
Da fins $c \in [a,b]$ s.a.

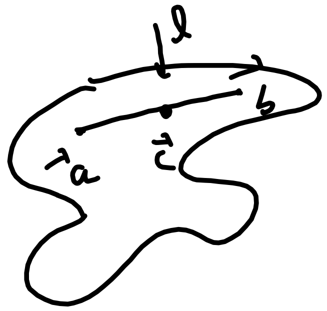
$$\frac{f(b) - f(a)}{b-a} = f'(c).$$



SETNING 3.2.3: La $A \subset \mathbb{R}^n$, la $f: A \to \mathbb{R}$ være deriverbar, og la $\vec{a}, \vec{b} \in A$ s.a. det rette linjestykket ℓ mellom $\vec{a}$ og $\vec{b}$ e i A. Da fins et punkt $\vec{c} \in \ell$ s.a.

$$f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \cdot (\vec{b} - \vec{a}).$$





$$f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \cdot (\vec{b} - \vec{a}).$$

Bevís: En parametriseing av linja l er $\vec{r}(t) = \vec{a} + t \cdot (\vec{b} - \vec{a})$, $t \in [0,1]$.

Observer: $\vec{r}\,'(t) = (\vec{b} - \vec{a})$

Definer $g(t) = f(\vec{r}(t))$.

Så nå er g en derivebar funksjon på $[0,1]$.

Fra Middelverdisetningen: $\exists\, t_0 \in [0,1]$

s.a. $$\frac{g(1) - g(0)}{1 - 0} = g'(t_0)$$

$$\Leftrightarrow \quad g(1) - g(0) = g'(t_0)$$

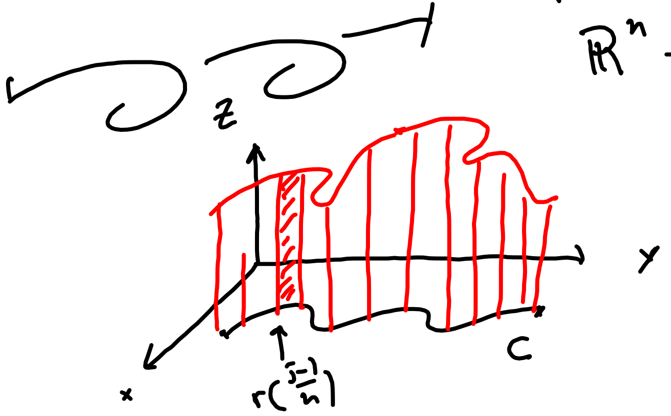
så $$f(b) - f(a) = g'(t_0).$$

Kjerneregel $g'(t_0) = f'(r(t_0)) \cdot r'(t_0)$
$= \nabla f(r(t_0)) \cdot (\vec{b} - \vec{a})$

Definer $\vec{c} = r(t_0)$, og vi er fremme. $\blacksquare$

Linjeintegraler for skalarfelter

$$\underbrace{\text{skalarfelter}}_{\mathbb{R}^n \to \mathbb{R}}$$



La $\vec{r} : [0,1] \to \mathbb{R}^2$ være en parametrisert kurve C.
Anta at $f(x,y)$ er kontinuerlig i nærheten av C.

$\int_C f$? Burde være arealet av det røde området over.

Arealet: $\lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} f\left(r\left(\frac{j-1}{n}\right)\right) \cdot \left\| r\left(\frac{j}{n}\right) - r\left(\frac{j-1}{n}\right) \right\|$

$0 \; \frac{1}{n} \; \frac{2}{n} \; \frac{3}{n} \cdots \; 1$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} f\left(r\left(\tfrac{j-1}{n}\right)\right) \cdot \left\| \frac{r\left(\frac{j}{n}\right) - r\left(\frac{j-1}{n}\right)}{1/n} \right\| \cdot 1/n$$

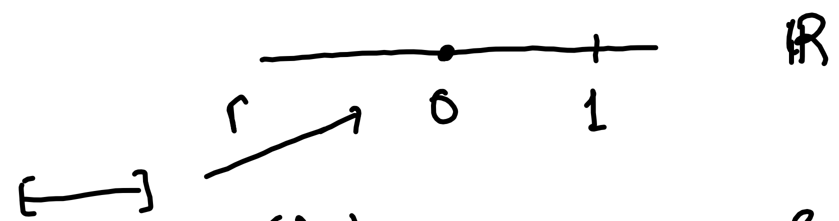
$$\approx \lim_{n\to\infty} \sum_{j=1}^{n} f\left(r\left(\tfrac{j-1}{n}\right)\right) \cdot \left\| r'\left(\tfrac{j-1}{n}\right) \right\| \cdot 1/n$$

Dette er Riemann-integralet til $f(r(t))|r'(t)|$.

DEFINERER: $$\int_C f \, ds := \int_0^1 f(r(t)) \cdot v(t) \, dt.$$

jan 27-11:30

Eks 1: La kurven C være intervallet $[0,1]$.



$0 \quad 1$ $\qquad r(t)=t.$ $\qquad$ Sett $f(x)=x^2$.

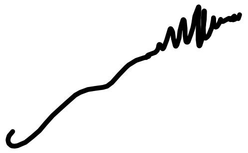
Sjekk:
$$\int_C f\,ds = \int_0^1 f(r(t))\cdot r'(t)\,dt$$
$$= \int_0^1 t^2\cdot 1\,dt = \frac{1}{3}.$$

Eks 2: Velg en annen parametrisering av C,

p.eks: $r(t)=t^7$.

$$\int_C f\,ds = \int_0^1 t^{14}\cdot 7\cdot t^6\,dt$$
$$= \int_0^1 7t^{20}\,dt = \left[\frac{7}{21}t^{21}\right]_0^1$$
$$= \underline{\underline{\frac{1}{3}}}.$$

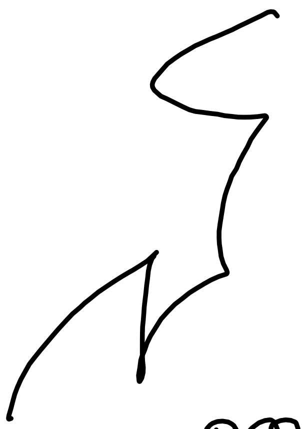
Glatt kurve: Vi sier at en parametrisert kurve $\vec{r}:[a,b]\to\mathbb{R}^n$ e <u>glatt</u> dersom den e kontinuerlig og deriverbar på (a,b).



$r:[a,b]\to\mathbb{R}^n$ sies å være <u>stykkevis glatt</u> om det fins

$a=a_0<a_1<a_2<\dots<a_m=b$

s.a. r e glatt på $[a_j,a_{j+1}]$.



<u>DEF:</u> La C være en stykkevis glatt parametrisert kurve $r:[a,b]\to\mathbb{R}^n$, og la $f:C\to\mathbb{R}$ være kont.

Da definerer vi

$$\int_C f\,ds := \int_a^b f(r(t))\cdot v(t)\,dt$$

(forutsatt at integralet eksisterer).

jan 27-12:00