Forelesning, 13. januar 2021

tirsdag 12. januar 2021 18:43

1. Kjerneregel for derivasjon
2. Linearisering (affinisering)

Kjerneregel

Eks. $f(x) = e^{x^2-1} = g(h(x))$

Ytre funksjon $g(u) = e^{u}$

Kjernen $u = h(x) = x^2 - 1$

Derivasjon: $f'(x) = e^{x^2-1} \cdot 2x$

$= g'(h(x)) \cdot h'(x)$

Alt. skrivemåte: $\frac{df}{dx} = \frac{dg}{du} \cdot \frac{dh}{dx}$

I flere variable:

$f = (f_1, \dots, f_n) : \mathbb{R} \to \mathbb{R}^n$

$t \mapsto (f_1(t), \dots, f_n(t))$

Gjennomgående eksempel:

$$f : t \mapsto (1, t, t^2) \in \mathbb{R}^3$$

Derivasjon:

$$f'(t) = (f_1'(t), \ldots, f_n'(t))$$

Eks. $f'(t) = (0, 1, 2t)$

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$(x_1, \ldots, x_n) \mapsto F(x_1, \ldots, x_n)$$

Eks. $F(x, y, z) = 4x + 2y - z^2$

Partiell derivasjon

$$\frac{\partial F}{\partial x} = 4 \quad \frac{\partial F}{\partial y} = 2 \quad \frac{\partial F}{\partial z} = -2z$$

$$F'(x, y, z) = \begin{pmatrix} 4 \\ 2 \\ -2z \end{pmatrix}$$

Sette sammen funksjonene:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[rr, bend right] & \mathbb{R}^n \arrow[r, "F"] & \mathbb{R} \end{tikzcd}$$

$$g(t) = F(f(t))$$

$$g(t) = F(1, t, t^2) = 4\cdot 1 + 2t - (t^2)^2$$
$$= 4 + 2t - t^4$$

$$g'(t) = \underline{\underline{2 - 4t^3}}$$

Ved å bruke kjerneregel:

$$g'(t) = \frac{d}{dt} F(f(t)) = F'(f(t)) \cdot f'(t)$$
$$= (4, 2, -2t^2) \cdot (0, 1, 2t) = 4\cdot 0 + 2\cdot 1 + (-2t^2)\cdot 2t$$
$$= \underline{\underline{2 - 4t^3}}$$

Nytt eksempel:

$$G, H : \mathbb{R}^2 \to \mathbb{R}^2 \qquad H(r, \theta) = (r\cos\theta, r\sin\theta)$$
$$F = G \circ H \qquad G(x, y) = (-y, x)$$

$$F(r, \theta) = G(H(r, \theta)) = (\underbrace{-r\sin\theta}_{F_1}, \underbrace{r\cos\theta}_{F_2})$$

$$\frac{\partial F_1}{\partial r} = -\sin\theta \qquad \frac{\partial F_1}{\partial \theta} = -r\cos\theta$$

$$\frac{\partial F_2}{\partial r} = \cos\theta \qquad \frac{\partial F_2}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$F'(r,\theta) = \begin{pmatrix} -\sin\theta & -r\cos\theta \\ \cos\theta & -r\sin\theta \end{pmatrix} \quad (*)$$

Jacobi-matriser

Kjerneregelen for $F = G \circ H$

$$F(r,\theta) = G\big(H(r,\theta)\big)$$

Derivasjon $\quad F'(r,\theta) = G'\big(H(r,\theta)\big) \cdot H'(r,\theta)$

$$H'(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$G'(x,y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

To Jacobi-matriser

$$F'(r,\theta) = G'(r\cos\theta, r\sin\theta) \cdot H'(r,\theta)$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} -\sin\theta & -r\cos\theta \end{pmatrix} \quad (*)$$

$$= \begin{pmatrix} \cos\theta & -r\sin\theta \end{pmatrix}$$

Gjennomgående eksempel:

$$f(t) = (1, t, t^2) \qquad F(x,y,z) = 4x + 2y - z^2$$

Regne ut Jacobi:

$$f'(t) = \begin{pmatrix} 0 \\ 1 \\ 2t \end{pmatrix} \qquad F'(x,y,z) = \begin{pmatrix} 4 & 2 & -2z \end{pmatrix}$$

$z = t^2$ $\quad$ $\cancel{2 = 2t}$ $\quad 4$

$$g'(t) = F'(f(t)) \cdot f'(t)$$

$$= \begin{pmatrix} 4 & 2 & -2(\cancel{2t}\, t^2) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2t \end{pmatrix} = \underline{\underline{2 - 4t^3}}$$

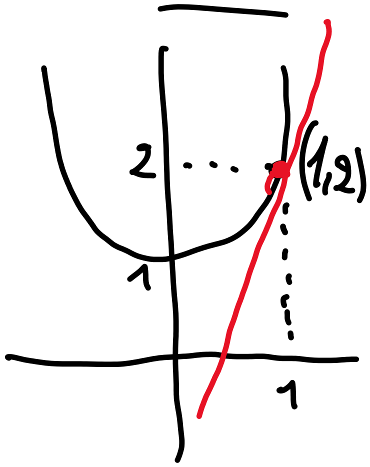
2.7.6 Kjerneregel på komponentform

$$H(\underline{x}) = F(G(\underline{x})) \qquad \underline{a} \in \mathbb{R}^n$$

$$\frac{\partial H_i}{\partial x_j} = \frac{\partial F_i}{\partial u_1}(G(\underline{a})) \cdot \frac{\partial G_1}{\partial x_j}(\underline{a}) + \cdots$$

$$+ \frac{\partial F_i}{\partial x_m}(G(\underline{a})) \cdot \frac{\partial G_m}{\partial x_j}(\underline{a})$$

2. Linearisering



$f(x) = x^2 + 1 \qquad f'(x) = 2x$

$f(1) = 2 \qquad f'(1) = 2$

Tangenten:

$$y - 2 = 2(x-1)$$

dvs $\quad y = 2x$

$$T_1 f(x) = 2x$$

$$y - f(1) = f'(1)(x-1)$$

$$y = T_1 f(x) = \underline{f(1) + f'(1)(x-1)}$$

$$\boxed{T_a f(x) = f(a) + f'(a)(x-a)}$$

Generaliserer til flere variable:

$$F(r,\theta) = (-r\sin\theta, r\cos\theta)$$

$$F'(r,\theta) = \begin{pmatrix} -\sin\theta & -r\cos\theta \\ \cos\theta & -r\sin\theta \end{pmatrix}$$

$$T_{(1,\frac{\pi}{4})} F(r,\theta) = F(1,\tfrac{\pi}{4}) + F'(1,\tfrac{\pi}{4}) \begin{pmatrix} r-1 \\ \theta - \frac{\pi}{4} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} + \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} r-1 \\ \theta - \frac{\pi}{4} \end{pmatrix}$$

Affin funksjon i (r, θ) som best approksimerer F i $(1, \frac{\pi}{4})$.

28.2 Defn. av linearisering

$F: \mathbb{R}^n \to \mathbb{R}^m$, $\underline{a} \in \mathbb{R}^n$

deriverbar i $\underline{x} = \underline{a}$

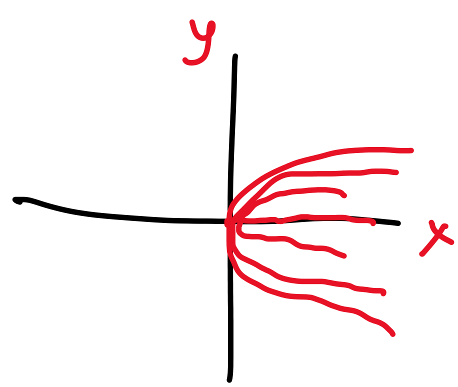
Linearisering til F i $\underline{a}$:

$$T_{\underline{a}} F(\underline{x}) = F(\underline{a}) + F'(\underline{a})(\underline{x} - \underline{a})$$

Eksempel (flate i $\mathbb{R}^3$)

$F: \mathbb{R}^2 \to \mathbb{R}^3$

$$F(x, y) = \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}$$



$$F'(x,y) = \begin{pmatrix} 2x & 0 \\ y & x \\ 0 & 2y \end{pmatrix}$$

Jacobi

Linearisering i $\underline{a} = (1,1)$

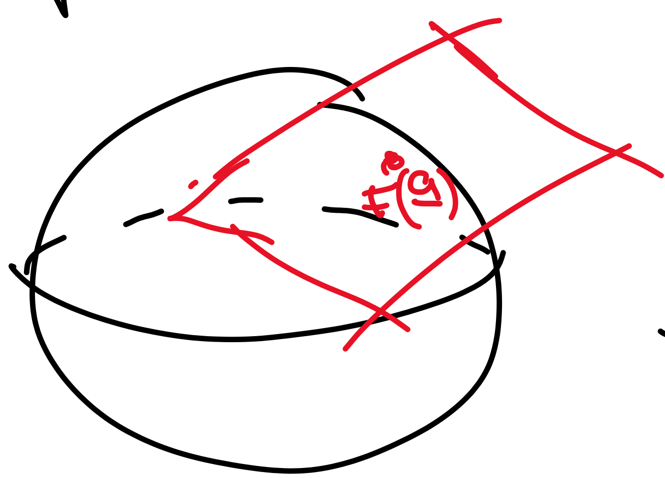
$$T_{(1,1)}F(x,y) = F(1,1) + F'(1,1)\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + 2(x-1) + 0(y-1) \\ 1 + 1(x-1) + 1(y-1) \\ 1 + 0(x-1) + 2(y-1) \end{pmatrix}$$

$$= \begin{pmatrix} 2x-1 \\ x+y-1 \\ 2y-1 \end{pmatrix}$$

Hvis flaten F var et kuleskall :

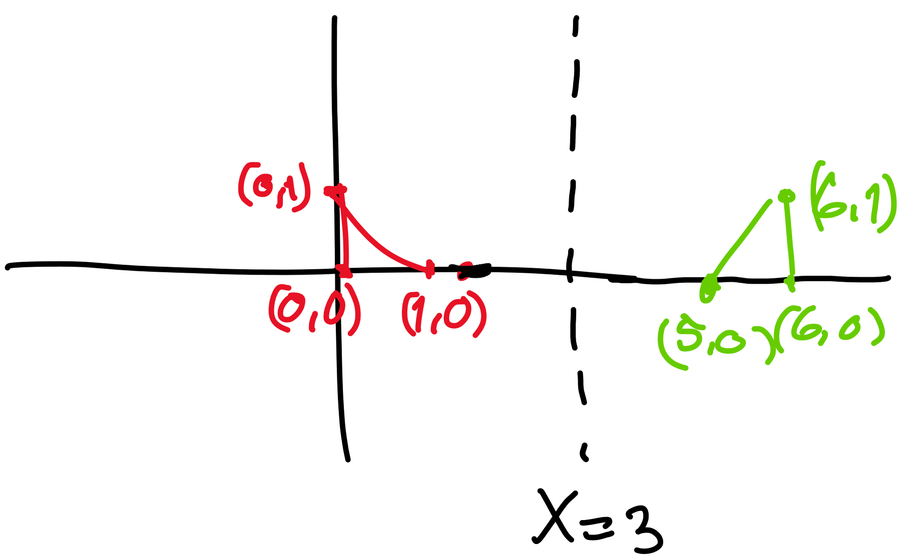


Et plan som tangerer flaten i punktet

$F(\underline{a})$.

Lineariseringen er den affinavbildningen som ligger tettest opp til funksjonen F i nærheten av punktet $\underline{a}$

$$\frac{1}{\|\underline{\varepsilon}\|}\left|F(\underline{a}+\underline{\varepsilon}) - T_{\underline{a}}F(\underline{a}+\underline{\varepsilon})\right| \xrightarrow[\underline{\varepsilon}\to 0]{} 0$$



$$F(x,y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$