Forelesning, 20. januar 2021

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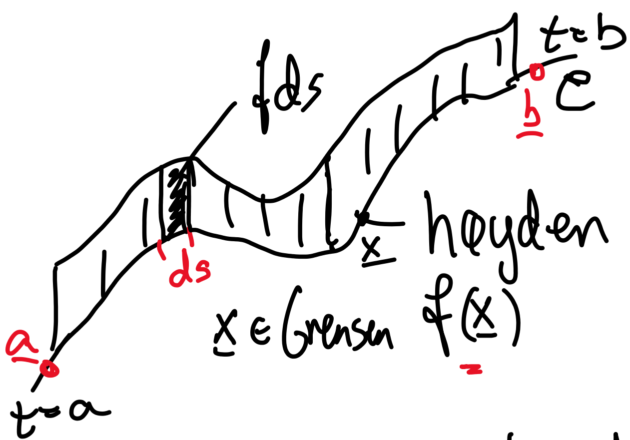
Linjeintegraler for skalar- og vektor-felt



$$\int_{\mathcal{C}} f\,ds = \text{areal av gjerdet}$$

$\mathcal{C}$: parametrisert kurve $\bar{r}(t) \in \mathbb{R}^2$

Vi har $f(\underline{x}) = f(\bar{r}(t))$

$ds = v \cdot dt \quad , v(t) = \|\bar{r}'(t)\|$

$$\boxed{\int_{\underline{a}}^{\underline{b}} f\,ds = \int_{\mathcal{C}} f\,ds = \int_a^b f(\bar{r}(t))\,\|\bar{r}'(t)\|\,dt}$$

Eks 1. $\mathcal{C}$: $\bar{r}(t) = (R\cos t, R\sin t, at) \quad 0 \le t \le \frac{\pi}{4}$

kurve

skalarfelt

$f(x,y,z) = x^2 - y^2$

$\bar{r}'(t) = (-R\sin t, R\cos t, a)$

$v(t) = \|\bar{r}'(t)\| = \sqrt{(-R\sin t)^2 + (R\cos t)^2 + a^2}$

$= \sqrt{R^2 + a^2}$

Linjeintegralet $\int_{\mathcal{C}} f\,ds = \int_0^{\frac{\pi}{4}} \left((R\cos t)^2 - (R\sin t)^2\right)\sqrt{R^2+a^2}\,dt$

$$\int_{\mathcal{C}} f\,ds = \int_0^{\frac{\pi}{4}} \left((R\cos t)^2 - (R\sin t)^2\right) \cdots$$

$$\boxed{\begin{aligned}&\cos 2t\\ &= \cos^2 t - \sin^2 t\end{aligned}}$$

$$= R^2\sqrt{R^2+a^2}\int_0^{\frac{\pi}{4}} \cos^2 t - \sin^2 t \; dt$$

$$= R^2\sqrt{R^2+a^2}\int_0^{\frac{\pi}{4}} \cos 2t \; dt$$

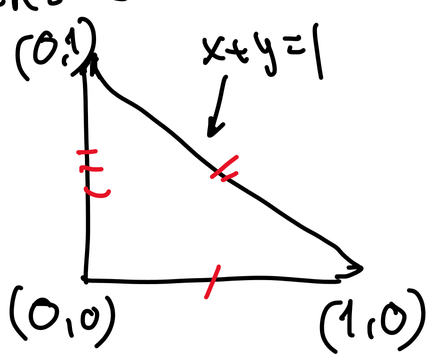
$u = 2t \quad dt = \frac{du}{2}$

$$= R^2\sqrt{R^2+a^2}\left[\frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{4}}$$

$$= R^2\sqrt{R^2+a^2}\left(\frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{2}\sin 0\right)$$

$$= \underline{\frac{1}{2}R^2\sqrt{R^2+a^2}}$$

Eks-2



$$\bar{r}(t) = \begin{cases} (t,0) & 0 \le t \le 1 \\ (2-t,\, t-1) & 1 \le t \le 2 \\ (0,\, 3-t) & 2 \le t \le 3 \end{cases}$$

$$\bar{r}'(t) = \begin{cases} (1,0) & 0 < t < 1 \\ (-1,1) & 1 < t < 2 \\ (0,-1) & 2 < t < 3 \end{cases}$$

$$f(x,y) = x+y$$

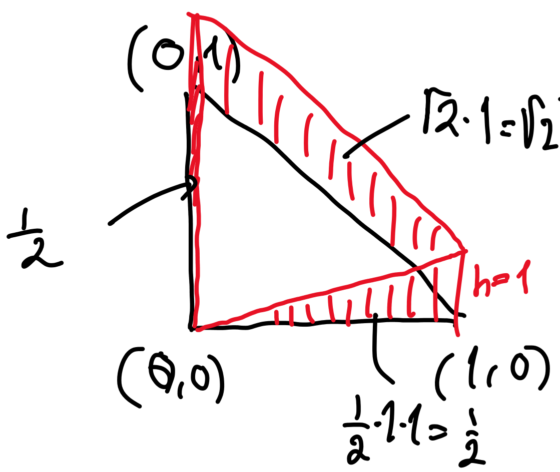
$$f(\bar{r}(t)) = \begin{cases} t & 0 \le t \le 1 \\ 1 & 1 \le t \le 2 \\ 3-t & 2 \le t \le 3 \end{cases} \qquad v(t) = \begin{cases} 1 & 0 < t < 1 \\ \sqrt{2} & 1 < t < 2 \\ 1 & 2 < t < 3 \end{cases}$$

$$\int_{\mathcal{C}} f\,ds = \int_0^1 t \cdot 1\, dt + \int_1^2 1\cdot\sqrt{2}\, dt + \int_2^3 (3-t)\, 1 \cdot dt$$

$$= \left[\frac{1}{2}t^2\right]_0^1 + \left[\sqrt{2}\cdot t\right]_1^2 + \left[3t - \frac{1}{2}t^2\right]_2^3$$

$$= \left(\frac{1}{2} - 0\right) + \left(2\sqrt{2} - \sqrt{2}\right) + \left(9 - \frac{9}{2} - 6 + \frac{4}{2}\right)$$

$$= \underline{\underline{\sqrt{2} + 1}}$$



$f(x,y) = x + y$

summen $= \underline{\underline{1 + \sqrt{2}}}$

Regneregler : 3.3.3 , 3.3.4

Ekvivalente parametriseringer

$\bar{r}_1 : [a,b] \longrightarrow \mathbb{R}^n$

$\bar{r}_2 : [c,d] \longrightarrow \mathbb{R}^n$

stykkevis glatte parametriseringer

$\bar{r}_1$ og $\bar{r}_2$ Ekvivalente

$$\begin{tikzcd} s \in [a,b] \arrow[rr, "\exists \varphi"] \arrow[dr, "\bar{r}_1"'] & \arrow[d, phantom, "\circ"] & {[c,d]} \ni t \arrow[dl, "\bar{r}_2"] \\ & \mathbb{R}^n & \end{tikzcd}$$

- $\bar{r}_2(\varphi(s)) = \bar{r}_1(s)$
- φ kontinuerlig
- φ surjektiv (verdimengden til φ $V_\varphi = [c,d]$)

φ strengt voksende

$\rightsquigarrow$ samme orientering

φ strengt avtagende
⇝ motsatt orientering

(– φ injektiv ($\varphi(s_1) = \varphi(s_2)$ betyr $s_1 = s_2$)
– $\varphi' \neq 0$ og φ' kontinuerlig

$$\boxed{\text{Linjeintegralet er uavhengig av valg av ekvivalente parametriseringer.}}$$

$$\int_{\mathcal{C}} f\,ds = \int_c^d f(\bar{r}_2(t))\, v_2(t)\,dt$$

$t = \varphi(s)$

$dt = \frac{d\varphi}{ds} \cdot ds$

$$= \int_a^b f(\underbrace{\bar{r}_2(\varphi(s))}_{\bar{r}_1(s)}) \cdot \underbrace{v_2(\varphi(s))\, \frac{d\varphi}{ds}}_{v_1(s)}\, ds$$

Mellomregning

$$\bar{r}_2(\varphi(s)) = \bar{r}_1(s)$$

$$\bar{r}_2{}'(\varphi(s)) \cdot \varphi'(s) = \bar{r}_1{}'(s) \qquad \text{Kjerneregel}$$

$$v_2(\varphi(s)) \cdot \frac{d\varphi}{ds} = v_1(s) \qquad \text{Abs. verdi}$$

$$= \int_a^b f(r_1(s)) \cdot v_1(s)\, ds$$

<u>Eks</u>

$$\begin{tikzcd} \theta \in [0, \frac{\pi}{2}] \arrow[rr, "\varphi(\theta) = \sin(\theta)"] \arrow[dr, "\bar{r}_1"'] & & {[0,1] \ni t} \arrow[dl, "\bar{r}_2"] \\ & \mathbb{R}^2 & \end{tikzcd}$$

$t = \sin\theta$

$\bar{r}_1(\theta) = (\cos\theta, \sin\theta)$ $\qquad$ $\bar{r}_2(t) = (\sqrt{1-t^2}, t)$

kvartsirkel

skalarfunksjonen: $f(x,y) = x+y$

$$\bar r_1(\theta) = (-\sin\theta, \cos\theta)$$

$$v_1(\theta) = 1$$

$$\int_C f\,ds = \int_0^{\frac{\pi}{2}} \cos\theta + \sin\theta \, d\theta$$

$$= \left[\sin\theta - \cos\theta\right]_0^{\frac{\pi}{2}}$$

$$= 1 - 0 - 0 + 1 = \underline{\underline{2}}$$

$$\bar r_2'(t) = \left(\frac{-t}{\sqrt{1-t^2}}, 1\right)$$

$$v_2(t) = \frac{1}{\sqrt{1-t^2}}$$

$$\int_C f\,ds = \int_0^1 \left(\sqrt{1-t^2} + t\right)\frac{1}{\sqrt{1-t^2}}\,dt$$

$$= \int_0^1 1 + \frac{t}{\sqrt{1-t^2}}\,dt$$

$$= \left[t - \sqrt{1-t^2}\right]_0^1$$

$$= 1 - 0 - 0 + 1 = \underline{\underline{2}}$$

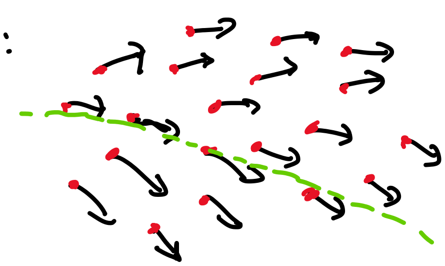
Linjeintegral for vektorfelt.

Hva er et vektorfelt?

1. svar: $\bar G : \mathbb{R}^2 \to \mathbb{R}^2$

2. svar: En funksjon som til et hvert punkt i planet tilordner en vektor i planet

3. svar:



(quiver)

$$\bar T(t) \cdot \bar G(\bar r(t)) = 0$$

Beveger oss på tvers av feltet (koster oss ingenting) felt ↓ | tangent Roster:

↓ ↓ behaglig!

$\bar{r}(t)$, $\bar{T}(t) = \frac{\bar{r}'(t)}{\|\bar{r}'(t)\|}$

kurve

$\bar{G}$ felt

$$\int_C \bar{G}\cdot d\bar{r} = \int_a^b \underbrace{\bar{T}(t)\cdot G(\bar{r}(t))}_{\text{skalarfelt}}\, dt$$

linjeintegralet av et vektorfelt.

Eks.

$\bar{G}(x,y) = (-y, x)$

(sirkulært felt)

$r(t) = \underline{a} + t(\underline{b} - \underline{a})$

$\underline{a} = (a_1, a_2)$ $t \in [0,1]$

$r'(t) = \underline{b} - \underline{a}$

$\bar{T}(t) = \frac{\underline{b}-\underline{a}}{\|\underline{b}-\underline{a}\|}$ ← antar for enkelhets skyld $\|\underline{b}-\underline{a}\| = 1$

$\underline{b} - \underline{a} = (b_1 - a_1, b_2 - a_2)$

$$\int_C \bar{G}\cdot d\bar{r} = \int_0^1 G(r(t))\cdot\bar{T}(t)\,dt$$

$$= \int_0^1 (-(a_2 + t(b_2 - a_2)), a_1 + t(b_1 - a_1)) \cdot \frac{\underline{b}-\underline{a}}{\|\underline{b}-\underline{a}\|}\,dt$$

$$= a_1 b_2 - a_2 b_1$$

$$\int_0^1 (-a_2 - t(b_2 - a_2), a_1 + t(b_1 - a_1))(b_1 - a_1, b_2 - a_2)\,dt$$

$$= \int_0^1$$

$$= \int_0 -a_2(b_1-a_1) - \cancel{t(b_2-a_2)(b_1-a_1)}$$

$$+ a_1(b_2-a_2) + \cancel{t(b_1-a_1)(b_2-a_2)}\, dt$$

$$= \int_0^1 -a_2 b_1 + \cancel{a_1 a_2} + a_1 b_2 - \cancel{a_1 a_2}\, dt$$

$$= \int_0^1 a_1 b_2 - a_2 b_1 \, dt = \underline{\underline{a_1 b_2 - a_2 b_1}}$$