

Oppgave 4, Eksamen 2023

$$\sum_{n=1}^{\infty} \frac{x^{n+2}}{n}$$

Forholdstesten: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+3}}{n+1}}{\frac{x^{n+2}}{n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} x \right| = |x|.$

V_i har absolutt konvergens for $x \in (-1, 1)$

$x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{n}$, alternierende, konvergent.

$x = 1$: $\sum_{n=1}^{\infty} \frac{1}{n}$, divergent.

konvergensområde: $[-1, 1)$.

Sett $S(x) = \sum_{n=1}^{\infty} \frac{x^{n+2}}{n} = x^2 \underbrace{\sum_{n=1}^{\infty} \frac{x^n}{n}}_{g(x)}$

$$g'(x) = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$$

integrer fra 0 til x

$$\int_0^x g'(t) dt = \left[-\ln(1-t) \right]_0^x$$

$$g(x) - \underbrace{g(0)}_0 = -\ln(1-x) \geq 0$$

$$g(x) = -\ln(1-x)$$

$$S(x) = x^2 g(x) = \underline{-x^2 \ln(1-x)}$$

Oppgave 3, eksamen 2022

a) Rekke $\sum_{n=0}^{\infty} \frac{1}{(n+1)2^n}$. Forholdstest: $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{1}{(n+2)2^{n+1}}}{\frac{1}{(n+1)2^n}} \right|$

$$= \left| \frac{n+1}{n+2} \cdot \frac{1}{2} \right| \rightarrow \frac{1}{2} < 1$$

Forholdsfesten sier at rekka konvergerer (absolutt)

b) Konvergensområdet til $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$: $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{x^{n+1}}{n+2}}{\frac{x^n}{n+1}} \right| = \left| \frac{n+1}{n+2} x \right|$

Rekka konvergerer absolutt for $x \in (-1, 1) \rightarrow |x|$

$x = 1$: Divergent

$x = -1$: alternerende, aufagende, går mot 0 $\Rightarrow$ konvergent.

Konvergensområdet er $[-1, 1)$

c) $S(x) = \sum_{n=0}^{\infty} \frac{x^n}{n+1}$

$$xS(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$(xS(x))' = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

integrer fra 0 til x .

$$xS(x) = -\ln(1-x)$$

$$S(x) = -\frac{1}{x} \ln(1-x)$$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)2^n} = S\left(\frac{1}{2}\right) = -\frac{1}{\frac{1}{2}} \ln\left(1 - \frac{1}{2}\right) = -2 \ln\left(\frac{1}{2}\right) = 2 \ln 2$$

Oppgave 1 Eksamen 2021

c) Rekke gitt ved $a_1 = x$, $a_{n+1} = -\frac{n+1}{n} \frac{x}{k} a_n$
konvergensradie?

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{-\frac{n+1}{n} \frac{x}{k} a_n}{a_n} \right| = \left| \frac{n+1}{n} \frac{x}{k} \right| \rightarrow \left| \frac{x}{k} \right|$$

Rekka konvergerer absolutt når $\left| \frac{x}{k} \right| < 1 \Leftrightarrow |x| < |k|$,
 $|k|$ er konvergensradien.

d) Vis $\sum_{n=1}^{\infty} (-1)^n \frac{n x^n}{2^{n-1}} = -\frac{4x}{(2+x)^2}$, når rekka konvergerer.

Løsning: Skriv som $x \sum_{n=1}^{\infty} (-1)^n \frac{n x^{n-1}}{2^{n-1}}$

$$\begin{aligned}
 \text{integrer } \int_0^x g(t) dt &= \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{2^{n-1}} = 2 \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{2^n} \\
 &= 2 \sum_{n=1}^{\infty} \left(-\frac{x}{2}\right)^n = 2 \frac{-\frac{x}{2}}{1 + \frac{x}{2}} = -\frac{2x+4-4}{x+2} \\
 &= -2 + \frac{4}{x+2}
 \end{aligned}$$

derivere

$$g(x) = -\frac{4}{(x+2)^2}$$

Derfor blir sammen $x \sum_{n=1}^{\infty} (-1)^n \frac{n x^{n-1}}{2^{n-1}} = x g(x) = -\frac{4x}{(x+2)^2}$

Oppgave 7 i quiz:

$$\sum_{n=1}^{\infty} \frac{x^{3n+1}}{n!}$$

$$\begin{aligned}
 \text{Forholdstest: } \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{\frac{x^{3(n+1)+1}}{(n+1)!}}{\frac{x^{3n+1}}{n!}} \right| \\
 &= \left| \frac{x^3}{n+1} \right| \rightarrow 0 \text{ når } n \rightarrow \infty, \text{ alle } x
 \end{aligned}$$

Rekka konvergerer derfor for alle x .

Eksamen 2023

Oppgave 1 $f(x,y) = 3y^2 - 3x^2 + 8xy - 6y - 8x$

a Stasjonære punkter: $\frac{\partial f}{\partial x} = -6x + 8y - 8 = 0$

$$\frac{\partial f}{\partial y} = 6y + 8x - 6 = 0$$

Vi må løse $-6x + 8y = 8$
 $8x + 6y = 6$

$$3 + \frac{16}{3} = \frac{28}{3}$$

$$\begin{pmatrix} -6 & 8 & 8 \\ 8 & 6 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{4}{3} & -\frac{4}{3} \\ 4 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{4}{3} & -\frac{4}{3} \\ 0 & \frac{28}{3} & \frac{28}{3} \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{4}{3} & -\frac{4}{3} \\ 0 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow x=0, y=1 \text{ eneste løsning.}$$

$(0,1)$ er eneste stasjonære punkt.

b) $\frac{\partial^2 f}{\partial x^2} = -6 \quad \frac{\partial^2 f}{\partial x \partial y} = 8 \quad \frac{\partial^2 f}{\partial y^2} = 6$

$$Hf(x,y) = \begin{pmatrix} -6 & 8 \\ 8 & 6 \end{pmatrix} \text{ for alle } x,y.$$

$D = -6 \cdot 6 - 8 \cdot 8 = -36 - 64 = -100 < 0$, derfor er $(0,1)$ sadelpunkt pga. andrederiverttesten.

Oppgave 2 maks/min for $f(x,y) = xy^2$ på $S = \{(x,y) : x^2 + y^2 = 1\}$:

Løsning: Definér $g(x,y) = x^2 + y^2 - 1 = 0$

$$\nabla f(x,y) = \begin{pmatrix} y^2 \\ 2xy \end{pmatrix} \quad \nabla g(x,y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$\nabla g = 0$ kun når $x=y=0$, men $(x,y)=(0,0)$ tilfredsstiller ikke $x^2 + y^2 = 1 \Rightarrow$ ingen kandidater der $\nabla g = \vec{0}$.

Vi må løse $\nabla f = \lambda \nabla g$:

$$\begin{pmatrix} y^2 \\ 2xy \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$$x^2 + y^2 = 1$$

Splitter opp i to muligheter:

$y=0$: siden $x^2 + y^2 = 1$, så må $x = \pm 1$
 $(\pm 1, 0)$ er kandidater

$y \neq 0$: Da må $x, \lambda \neq 0$ pga. likning 1.

$$\left. \begin{array}{l} y^2 = 2\lambda x \\ 2xy = 2\lambda y \end{array} \right\} \quad \left. \begin{array}{l} y^2 = 2\lambda x \\ x = \lambda \end{array} \right\} \quad y^2 = 2x^2$$

$$x^2 + y^2 = 1 \Rightarrow x^2 + 2x^2 = 1 \Rightarrow 3x^2 = 1 \Rightarrow x = \pm \sqrt{\frac{1}{3}}$$

$$y = \pm \sqrt{\frac{2}{3}}$$

x, y^2 $(\pm \sqrt{\frac{1}{3}}, \pm \sqrt{\frac{2}{3}})$ er (4) kandidater

$$f(\pm 1, 0) = 0$$

$$f(\sqrt{\frac{1}{3}}, \pm \sqrt{\frac{2}{3}}) = \sqrt{\frac{1}{3}} \cdot \frac{2}{3} \quad \text{størst (maks)}$$

$$f(-\sqrt{\frac{1}{3}}, \pm \sqrt{\frac{2}{3}}) = -\sqrt{\frac{1}{3}} \cdot \frac{2}{3} \quad \text{minst (min)}$$

Vi må ha maks/min, siden S er lukket/begrenset og f kontinuert.

Oppgave 3

$$A = \begin{pmatrix} 0.4 & 0.2 \\ -0.3 & 1.1 \end{pmatrix}$$

$$a) \det(\lambda I - A) = \begin{vmatrix} \lambda - 0.4 & -0.2 \\ 0.3 & \lambda - 1.1 \end{vmatrix} = (\lambda - 0.4)(\lambda - 1.1) + 0.06$$

$$= \lambda^2 - 1.5\lambda + 0.5$$

$$\lambda = \frac{1.5 \pm \sqrt{2.25 - 2}}{2} = \frac{1.5 \pm 0.5}{2} \Rightarrow \lambda_1 = 1, \lambda_2 = \frac{1}{2}$$

$$\lambda_1 = 1: I - A = \begin{pmatrix} 0.6 & -0.2 \\ 0.3 & -0.1 \end{pmatrix} \sim \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix} \quad \text{; egenvektor: } 3x - y = 0$$

$$\Rightarrow \underline{(1, 3) \text{ egenvektor}}$$

$$\lambda_2 = \frac{1}{2}: \frac{1}{2}I - A = \begin{pmatrix} 0.1 & -0.2 \\ 0.3 & -0.6 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \Rightarrow \underline{(2, 1) \text{ egenvektor}}$$

b) $\vec{v} = (-5, 0)$. Uttrykk denne med egenvektorene over:

$$\begin{pmatrix} 1 & 2 & -5 \\ 3 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 \\ 0 & -5 & 15 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 \\ 0 & 1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \end{pmatrix}$$

$$\Rightarrow \vec{v} = 1 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\lim_{n \rightarrow \infty} A^n \vec{v} = \lim_{n \rightarrow \infty} A^n \left(1 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) = \lim_{n \rightarrow \infty} \left(1^n \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 3 \cdot \left(\frac{1}{2} \right)^n \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right)$$

$$= \underline{\underline{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}}$$

$\rightarrow 0$

