

Forelesning 21. mai

Eksamen 2023

Oppgave 5 $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 6 \end{pmatrix}$

a) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 6 \end{pmatrix} \xrightarrow[\sim]{\substack{\text{II} - 2\text{I} \\ \text{III} - 3\text{I}}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{pmatrix} \xrightarrow[\sim]{\substack{\text{III} - \text{II} \\ \text{II} \cdot (-\frac{1}{3})}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{I} - 2\text{II}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

b) For hvilke $\vec{b}$ har $A\vec{x} = \vec{b}$ ingen løsning? Samme radoperasjoner som over på $[A \ \vec{b}]$ må da gi

$$\begin{pmatrix} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 0 & t \end{pmatrix}, \text{ der } t \neq 0.$$

Inverse radoperasjoner:

$$\begin{pmatrix} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 0 & t \end{pmatrix} \xrightarrow{\text{I} + 2\text{II}} \begin{pmatrix} 1 & 2 & 3 & a+2b \\ 0 & 1 & 1 & b \\ 0 & 0 & 0 & t \end{pmatrix} \xrightarrow[\sim]{\substack{\text{II} \cdot (-3) \\ \text{III} + \text{II}}} \begin{pmatrix} 1 & 2 & 3 & a+2b \\ 0 & -3 & -3 & -3b \\ 0 & -3 & -3 & t-3b \end{pmatrix}$$

$$\xrightarrow[\sim]{\substack{\text{III} + 3\text{I} \\ \text{II} + 2\text{I}}} \begin{pmatrix} 1 & 2 & 3 & a+2b \\ 2 & 1 & 3 & 2a+b \\ 3 & 3 & 6 & t+3a+3b \end{pmatrix}$$

$\vec{b}$

$\vec{b}$ må altså være på formen

$$\begin{pmatrix} a+2b \\ 2a+b \\ t+3a+3b \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\neq 0$

Oppgave 6

$$\vec{F}(x,y) = (\underbrace{y \cos(xy)}_{\vec{F}_1} - \underbrace{2xy e^{x^2 y}}_{\vec{F}_2}, \underbrace{x \cos(xy)}_{\vec{F}_1} - \underbrace{x^2 e^{x^2 y}}_{\vec{F}_2} + \underbrace{x}_{\vec{F}_2})$$

Er feltet konservativt?

$$\frac{\partial \phi}{\partial x} = \underbrace{y \cos(xy) - 2xye^{x^2y}}_{\phi(x,y)} - y \Rightarrow \phi(x,y) = \underbrace{\sin(xy) - e^{x^2y}}_{\phi(x,y)} - yx + C_1(y)$$

$$\frac{\partial \phi}{\partial y} = \underbrace{x \cos(xy) - x^2 e^{x^2y}}_{\phi(x,y)} + x \Rightarrow \phi(x,y) = \underbrace{\sin(xy) - e^{x^2y}}_{\phi(x,y)} + yx + C_2(x)$$

$$\text{sett } \vec{F}_1 = (y \cos(xy) - 2xye^{x^2y}, x \cos(xy) - x^2 e^{x^2y})$$

$$\vec{F}_2 = (-y, x).$$

$$\text{Da er } \vec{F} = \vec{F}_1 + \vec{F}_2, \text{ og vi ser at } \vec{F}_1 = \nabla \phi, \text{ der } \phi(x,y) = \sin(xy) - e^{x^2y}$$

siden C er lukket kurve:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (\vec{F}_1 + \vec{F}_2) \cdot d\vec{r} = \underbrace{\int_C \vec{F}_1 \cdot d\vec{r}}_0 + \int_C \vec{F}_2 \cdot d\vec{r}$$

$$\begin{aligned} &= \int_C \vec{F}_2 \cdot d\vec{r} = \int_0^{2\pi} (-y, x) \cdot (-\sin t, \cos t) dt \\ &= \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t) dt \\ &= \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = \int_0^{2\pi} 1 dt = \underline{\underline{2\pi}} \end{aligned}$$

siden $\vec{F}_1$ konservativ,
 C lukket.

$\vec{r}(t) = (\cos t, \sin t)$
 $\vec{r}'(t) = (-\sin t, \cos t)$

Eksamen 2022

$$1) A = \begin{pmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{pmatrix}$$

$$\begin{aligned} a) \det(\lambda I - A) &= (\lambda - 0.3)(\lambda - 0.4) - 0.42 = \lambda^2 - 0.7\lambda - 0.3 \\ \lambda &= \frac{0.7 \pm \sqrt{0.49 + 1.2}}{2} = \frac{0.7 \pm 1.3}{2} \Rightarrow \lambda_1 = -0.3 \quad \lambda_2 = 1 \end{aligned}$$

Egenvektor $\lambda_1 = -0.3$:

$$-0.3 I - A = \begin{pmatrix} -0.6 & -0.6 \\ -0.7 & -0.7 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \underline{(1, -1)} \text{ sps. egenvek.}$$

$$x + y = 0 \Rightarrow y = -x$$

Egenvektor $\lambda_2 = 1$:

$$I - A = \begin{pmatrix} 0.7 & -0.6 \\ -0.7 & 0.6 \end{pmatrix} \sim \begin{pmatrix} 7 & -6 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} 7x - 6y = 0 \\ 6x - 7y = 0 \end{matrix} \Rightarrow \underline{(6, 7)} \text{ spesiell egenvektor.}$$

b) $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Hva blir $\lim_{n \rightarrow \infty} A^n \vec{v}$?

$$\begin{pmatrix} 1 & 6 & 1 \\ -1 & 7 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 6 & 1 \\ 0 & 13 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 6 & 1 \\ 0 & 1 & \frac{2}{13} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{1}{13} \\ 0 & 1 & \frac{2}{13} \end{pmatrix}$$

$$\lim_{n \rightarrow \infty} A^n \vec{v} = \lim_{n \rightarrow \infty} \left(A^n \left(\frac{1}{13} \vec{v}_1 + \frac{2}{13} \vec{v}_2 \right) \right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{1}{13} \cdot (-0.3)^n \vec{v}_1 + \frac{2}{13} \cdot 1^n \vec{v}_2 \right)$$

$$= \frac{2}{13} \vec{v}_2 = \underline{\underline{\begin{pmatrix} 6 \\ 7 \end{pmatrix}}}$$

Oppgave 2 $f(x, y) = 2xy^2 + 4xy - x^2 + 1$

a) Stasjonære punkter: $\frac{\partial f}{\partial x} = 2y^2 + 4y - 2x = 0$

$$\frac{\partial f}{\partial y} = 4xy + 4x = 0 :$$

ser først på $x=0$: Da er $\frac{\partial f}{\partial y} = 0$, og $\frac{\partial f}{\partial x} = 2y^2 + 4y = 2y(y+2) = 0$ når $y=0$ eller $y=-2$.

$(0, 0)$ og $(0, -2)$ er stasjonære

ser så på $x \neq 0$: $\frac{\partial f}{\partial y} = 4x(y+1) = 0$ bare når $y = -1$.

$\frac{\partial f}{\partial x} = 0$ når $2 - 4 - 2x = 0 \Rightarrow 2x = -2 \Rightarrow x = -1$
 $(-1, -1)$ er stasjonært.

$$b) \quad \frac{\partial^2 f}{\partial x^2} = -2 \quad \frac{\partial^2 f}{\partial x \partial y} = 4y+4 \quad \frac{\partial^2 f}{\partial y^2} = 4x$$

$$Hf(x,y) = \begin{pmatrix} -2 & 4y+4 \\ 4y+4 & 4x \end{pmatrix}$$

$$Hf(0,0) = \begin{pmatrix} -2 & 4 \\ 4 & 0 \end{pmatrix} \Rightarrow D = -16 < 0 \Rightarrow \text{sadelpunkt}$$

$$Hf(0,-2) = \begin{pmatrix} -2 & -4 \\ -4 & 0 \end{pmatrix} \Rightarrow D = -16 < 0 \Rightarrow \text{sadelpunkt}$$

$$Hf(-1,-1) = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix} \Rightarrow D = 8 > 0, \lambda = -2 < 0 \Rightarrow \text{lokalt maks.}$$

c) Minste verdi til f under betingelsen $g(x,y) = x - y^2 - 2y = 0$

$$\nabla f = \begin{pmatrix} 2y^2 + 4y - 2x \\ 4xy + 4x \end{pmatrix} \quad \nabla g = \begin{pmatrix} 1 \\ -2y - 2 \end{pmatrix}$$

det viser seg at de tre punktene i b) også tilfredsstiller betingelsen.

$$\nabla g = \vec{0} \text{ umulig} \Rightarrow \text{ingen kandidater herfra.}$$

$$\nabla f = \lambda \nabla g: \quad \begin{aligned} 2y^2 + 4y - 2x &= \lambda \\ 4xy + 4x &= \lambda(-2y - 2) \\ x - y^2 - 2y &= 0 \end{aligned}$$

$$\text{Likning 2: } 4x(y+1) = -2\lambda(y+1)$$

$$y \neq -1: \text{Likning 2 kan forkortes til } 4x = -2\lambda \Rightarrow \lambda = -2x.$$

$$\text{Dette gir, innsett i likning 1: } \left. \begin{aligned} 2y^2 + 4y &= 0 \\ x - y^2 - 2y &= 0 \end{aligned} \right\} \left. \begin{aligned} 2y(y+2) &= 0 \\ x &= y^2 + 2y \end{aligned} \right\} \begin{aligned} y=0 \text{ eller } y=-2 \\ \downarrow \quad \downarrow \\ x=0 \quad x=0 \end{aligned}$$

$$\text{Dette gir kandidatene: } (0,0), (0,-2)$$

$$y = -1: \text{Likning 2 er da alltid oppfylt. Likning 1 også ok}$$

$$x - y^2 - 2y = 0 \Rightarrow x = y^2 + 2y = 1 - 2 = -1. \text{ Dette gir } (-1,-1) \text{ som kandidat}$$

$$f(0,0) = 1 \quad \searrow \text{min. begge to.}$$

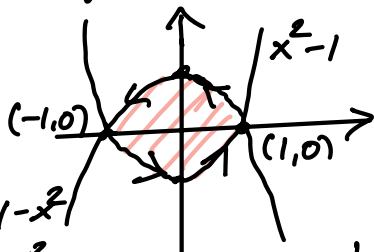
$$f(0,-2) = 1 \quad \searrow$$

$$f(-1, -1) = 2$$

Hva skjer når $x, y \rightarrow \infty$? $2xy^2 + 4xy - x^2 + 1 \rightarrow \infty$ når $y \rightarrow \infty$.
(innsett $x = y^2 + 2y$)

Oppgave 4

a) Området avgrenset av $y = x^2 - 1$ og $y = 1 - x^2$



$$\begin{aligned} A &= \int_{-1}^1 \left[\int_{x^2-1}^{1-x^2} 1 \, dy \right] dx = \int_{-1}^1 (1-x^2 - (x^2-1)) dx = \int_{-1}^1 (2-2x^2) dx \\ &= \left[2x - \frac{2}{3}x^3 \right]_{-1}^1 = 2 - \frac{2}{3} + 2 - \frac{2}{3} \\ &= 4 - \frac{4}{3} = \underline{\underline{\frac{8}{3}}} \end{aligned}$$

b) $\vec{F}(x, y) = \overbrace{(y^3 - y + x)}^P \vec{i} + \overbrace{(2x + y^2 + 3xy^2)}^Q \vec{j}$

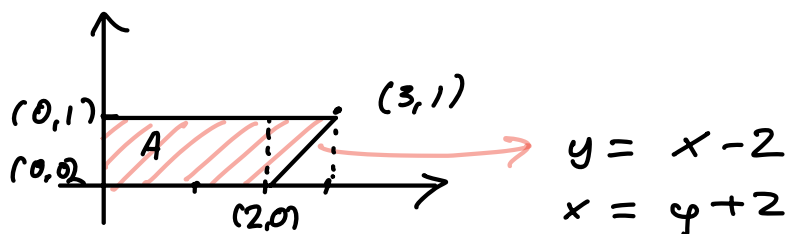
$\mathcal{C}$ randen til området A fra a), orientert mot klokke

Hva blir $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$?

Greens teorem: $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2 + 3y^2 - (3y^2 - 1) = 3$

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \iint_A 3 \, dx \, dy = 3 \iint_A 1 \, dx \, dy = 3 \cdot \frac{8}{3} = \underline{\underline{8}}$$

Eksamen 2021



$$\begin{aligned} y &= x - 2 \\ x &= y + 2 \end{aligned}$$

a) massemidelpunkt?

$$\iint_A x dx dy = \int_0^1 \left[\int_0^{y+2} x dx \right] dy = \int_0^1 \left[\frac{1}{2} x^2 \right]_0^{y+2} dy = \int_0^1 \frac{1}{2} (y+2)^2 dy$$

$$= \left[\frac{1}{6} (y+2)^3 \right]_0^1 = \frac{1}{6} (27 - 8) = \frac{19}{6}$$

$$\iint_A y dx dy = \int_0^1 \left[\int_0^{y+2} y dx \right] dy = \int_0^1 [x y]_0^{y+2} dy = \int_0^1 (y^2 + 2y) dy$$

$$= \left[\frac{1}{3} y^3 + y^2 \right]_0^1 = \frac{1}{3} + 1 = \frac{4}{3}$$

$$\text{area} = \iint_A 1 dx dy = 2.5 = \frac{5}{2}$$

$$\bar{x} = \frac{\iint_A x dx dy}{\iint_A 1 dx dy} = \frac{\frac{19}{6}}{\frac{5}{2}} = \frac{19}{15}$$

$$\bar{y} = \frac{\iint_A y dx dy}{\iint_A 1 dx dy} = \frac{\frac{4}{3}}{\frac{5}{2}} = \frac{8}{15}$$

$(\frac{19}{15}, \frac{8}{15})$ Mittelpunkt.

b) $\vec{r}(x,y) = (x, y, \overbrace{1 + \frac{1}{4}x + \frac{1}{2}y}^{f(x,y)})$, $0 \leq y \leq 1$, $0 \leq x \leq y+2$

$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} \vec{e} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x} \vec{e} - \frac{\partial f}{\partial y} \vec{j} + \vec{k}$$

$$\left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{\frac{1}{16} + \frac{1}{4} + 1}$$

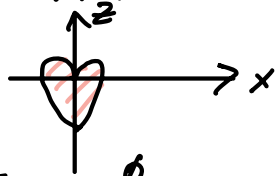
$$= \sqrt{\frac{1+y+16}{16}} = \frac{\sqrt{21}}{4}$$

$$A = \int_0^1 \left[\int_0^{y+2} \left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| dx \right] dy$$

$$= \int_0^1 \left[\int_0^{y+2} \frac{\sqrt{21}}{4} dx \right] dy = \frac{\sqrt{21}}{4} A(A) = \frac{\sqrt{21}}{4} \cdot \frac{5}{2} = \underline{\underline{\frac{5}{8} \sqrt{21}}}$$

Oppgave 4 eksamen 2020

Volumet av området avgrenset av $\rho = \phi$; kulekoordinater:



$$\begin{aligned} V &= \int_0^{2\pi} \left[\int_0^{\pi} \left[\int_0^{\phi} \rho^2 \sin \phi \, d\rho \right] d\phi \right] d\theta \\ &= \int_0^{2\pi} \left[\int_0^{\pi} \frac{1}{3} \phi^3 \sin \phi \, d\phi \right] d\theta = \frac{2\pi}{3} \int_0^{\pi} \phi^3 \sin \phi \, d\phi \\ &= \frac{2\pi}{3} \left(\left[-\phi^3 \cos \phi \right]_0^{\pi} + \int_0^{\pi} 3\phi^2 \cos \phi \, d\phi \right) \\ &= \frac{2\pi}{3} \left(\pi^3 + 3 \left(\left[\phi^2 \sin \phi \right]_0^{\pi} - \int_0^{\pi} 2\phi \sin \phi \, d\phi \right) \right) \\ &= \frac{2\pi}{3} \left(\pi^3 - 6 \left(\left[-\phi \cos \phi \right]_0^{\pi} + \int_0^{\pi} \cos \phi \, d\phi \right) \right) \\ &= \frac{2\pi}{3} (\pi^3 - 6\pi) = \underline{\underline{\frac{2\pi^2}{3} (\pi^2 - 6)}} \end{aligned}$$