

Eksempel Torus. $A = [0, 2\pi]^2$ $u, v \in [0, 2\pi]$,

$$r(u, v) = (R + r \cos u) \cos v \underline{i} + (R + r \cos u) \sin v \underline{j} + r \sin u \underline{k}.$$

(Seksjon 3.9).

$$\frac{\partial r}{\partial u} = -r \sin u \cos v \underline{i} - r \sin u \sin v \underline{j} + r \cos u \underline{k}$$

$$\frac{\partial r}{\partial v} = -(R + r \cos u) \sin v \underline{i} + (R + r \cos u) \cos v \underline{j}$$

R, r radius

$$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \cancel{r(R + r \cos u)}$$



$$\begin{vmatrix} \underline{i} & -r \sin u \cos v & -(R + r \cos u) \sin v \\ \underline{j} & -r \sin u \sin v & (R + r \cos u) \cos v \\ \underline{k} & r \cos u & 0 \end{vmatrix}$$

$$= -r(R + r \cos u) \begin{bmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{bmatrix}$$

$$\Rightarrow \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| = r(R + r \cos u).$$

$$\text{Arealet til torusen} = \iint_A \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| du dv$$

$$= \int_0^{2\pi} \left[\int_0^{2\pi} r(R + r \cos u) dv \right] du$$

$$= 2\pi \int_0^{2\pi} r(R + \cos u) du$$

$$= 4\pi^2 r R, \quad \left(\text{fordi } \int_0^{2\pi} \cos u du = 0 \right).$$

Spesialtilfellet:

$$r(x, y) = \begin{bmatrix} x \\ y \\ f(x, y) \end{bmatrix} \quad (x, y) \in A.$$

$$\frac{\partial r}{\partial x} = \begin{bmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x} \end{bmatrix}, \quad \frac{\partial r}{\partial y} = \begin{bmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} &= \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} \\ &= \begin{bmatrix} -\frac{\partial f}{\partial x} \\ -\frac{\partial f}{\partial y} \\ 1 \end{bmatrix} \Rightarrow \left| \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} \right| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}. \end{aligned}$$

Arealen til funksjonsgrafen $z = f(x, y)$ over A er dermed

$$\text{Areal} = \iint_A \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dx dy.$$

Flateintegraler av skalarfelt

$(u, v) \in A$ $r: A \rightarrow \mathbb{R}^3$ γ La $T \subseteq \mathbb{R}^3$ være flaten beskrevet av A og r . og at massetettheten i punktet $(x, y, z) \in T$ er $f(x, y, z)$. Da er massen til T ,

$$m(T) = \iint_A f(r(u, v)) \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| du dv$$

Kalles flateintegralen til f over T .

Øks. 6.5. Green's teorem

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En viktig sammenheng mellom linjeintegraler for vektorfelt og dobbeltintegraler.

Anta at $\underline{F}(x,y) = (P(x,y), Q(x,y))$ er et vektorfelt.

La C være en kurve parametrisert ved $r: [a,b] \rightarrow \mathbb{R}^2$, $r(t) = (x(t), y(t))$, $t \in [a,b]$. Husk at vi definerer linjeintegralet:

$$\begin{aligned} \int_C \underline{F} \cdot d\underline{r} &:= \int_a^b \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t) dt \\ &= \int_a^b \underline{F}(x(t), y(t)) \cdot (x'(t), y'(t)) dt \\ &= \int_a^b P(x(t), y(t)) x'(t) + Q(x(t), y(t)) y'(t) dt \end{aligned}$$

vi skriver dette som $\int_C P dx + Q dy$.

Teorem 6.5.1 (Greens teorem)

Anta at C er enkel og lukket ($\underline{r}(a) = \underline{r}(b)$) enkel = "har ingen selvskjæring".

med en stykkevis glatt parametrisering $\underline{r}$.

La R være området begrenset av C .

Anta de partielle deriverte til P og Q er kontinuerlige i et åpent område som inneholder R .

Da er

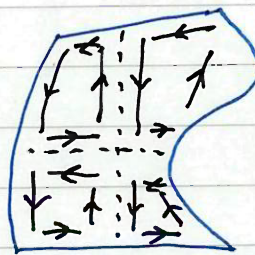
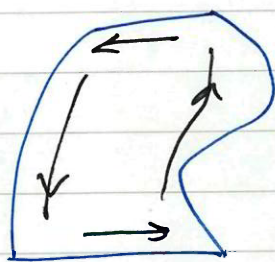
$$\int_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

der C er orientert mot klokken.

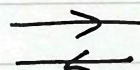
Bevis
(Leene)

Dele i enklere områder:

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Vi får kansellering
i linjeintegraler



Nok å vise at både

$$\int_C P dx = - \iint_R \frac{\partial P}{\partial y} dx dy$$

$$\text{og } \int_C Q dy = \iint_R \frac{\partial Q}{\partial x} dx dy.$$

Start med Type I og II områder.

Greens teorem kan brukes begge veier.

Beregne arealer:

(Korollar
$$\text{Areal}(R) = \int_C x dy = - \int_C y dx = \frac{1}{2} \int_C -y dx + x dy.$$

Bevis.

Velg: (i) $\underline{F}(x,y) = (0, x)$ $\Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 0 = 1$
 (ii) $\underline{F}(x,y) = (-y, 0)$ $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 + 1 = 1$
 (iii) $\underline{F}(x,y) = (-\frac{1}{2}y, \frac{1}{2}x)$ $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{1}{2} + \frac{1}{2} = 1.$

Da er
$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \iint_R 1 \cdot dx dy = \text{Areal}(R)$$

i alle tre tilfellene.

(Eksempel Hva er arealet til ellipsen $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$?

En parametrisering er $\underline{r}(t) = (a \cos t, b \sin t)$,
 $0 \leq t \leq 2\pi$. Denne er orientert mot klokken.

Kan bruke Formelen,

$$\text{Areal}(R) = \frac{1}{2} \int_C -y dx + x dy \quad \left(\begin{array}{l} x = a \cos t \\ y = b \sin t \end{array} \right)$$

$$= \frac{1}{2} \int_0^{2\pi} (-y x' + x y') dt$$

$$= \frac{1}{2} \int_0^{2\pi} (-b \sin t)(-a \sin t) + a \cos t(b \cos t) dt$$

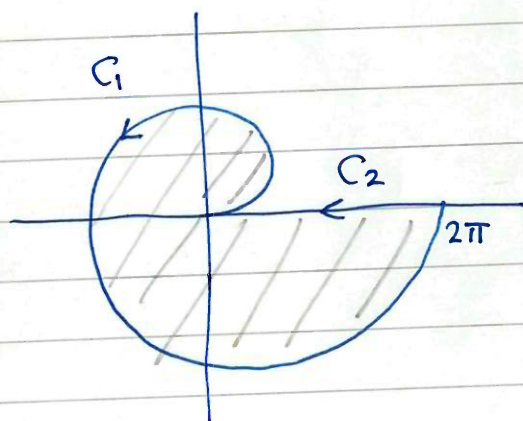
$$= \frac{ab}{2} \int_0^{2\pi} \sin^2 t + \cos^2 t dt$$

$$= \frac{ab}{2} \int_0^{2\pi} 1 \cdot dt$$

$$= \pi ab.$$

Eksempel Hva blir arealet avgrenset av
 $r(t) = (t \cos t, t \sin t)$, $0 \leq t \leq 2\pi$, og den
 positive x -aksen?

Løsning



$$\begin{aligned} \underline{r}_1(t) &= (t \cos t, t \sin t), \\ \underline{r}_2(t) &= (2\pi - t, 0), \\ 0 &\leq t \leq 2\pi. \end{aligned}$$

R er avgrenset av C_1 og C_2 , C_2 går fra $(2\pi, 0)$ til $(0, 0)$.
 kan bruke formelen

$$A = \oint_C x dy = \int_{C_1} x dy + \int_{C_2} x dy, \quad \begin{matrix} x = t \cos t \\ y = t \sin t \end{matrix}$$

$$\int_{C_1} x dy = \int_0^{2\pi} t \cos t (y'(t)) dt$$

$$= \int_0^{2\pi} t \cos t (\sin t + t \cos t) dt$$

$$= \frac{1}{2} \int_0^{2\pi} t \sin 2t dt + \int_0^{2\pi} t^2 \cos^2 t dt$$

$$= \frac{1}{2} \int_0^{2\pi} t \sin 2t dt + \int_0^{2\pi} t^2 \left(\frac{\cos 2t + 1}{2} \right) dt$$

Delvis integrasjon: $\int_0^{2\pi} t^2 \cos 2t dt = \left[t^2 \frac{\sin 2t}{2} \right]_0^{2\pi} - \int_0^{2\pi} 2t \cdot \frac{\sin 2t}{2} dt$

$$= - \int_0^{2\pi} t \sin 2t dt$$

Dermed er $\int_{C_1} x dy = \frac{1}{2} \int_0^{2\pi} t^2 dt = \frac{1}{2} \left[\frac{t^3}{3} \right]_0^{2\pi} = \frac{4}{3} \pi^3$.

$$\int_{C_2} x dy = \int_0^{2\pi} (2\pi - t) y'(t) dt = \int_0^{2\pi} (2\pi - t) \cdot 0 dt = 0$$

Da er $\oint_C x dy = \frac{4}{3} \pi^3$.