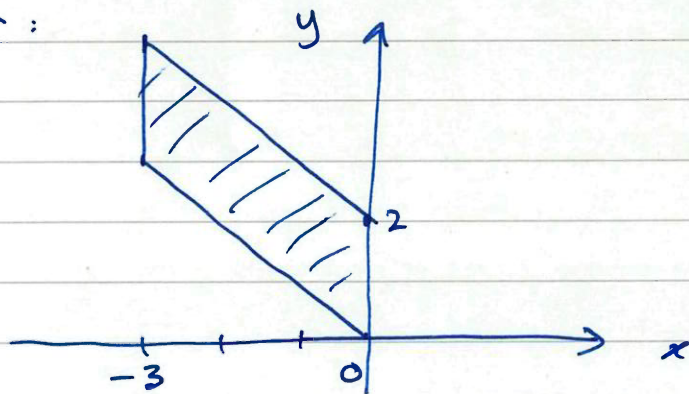


Trüppelintegraler Eksempel: Regn ut $\iiint_S x \, dx \, dy \, dz$

når S er området avgrenset av S planene $y = -x$, $y = -x+2$, $x = -3$, $x = 0$, under ellipsoiden $z = x^2 + 2y^2$ og over planet $z = 0$.

Løsning Figur:



$$S = \{ (x, y, z) \in \mathbb{R}^3 : -3 \leq x \leq 0, -x \leq y \leq -x+2, 0 \leq z \leq x^2 + 2y^2 \}$$

$$I = \int_{-3}^0 \left[\int_{-x}^{-x+2} \left[\int_0^{x^2+2y^2} x \, dz \right] dy \right] dx$$

$$= \int_{-3}^0 \int_{-x}^{-x+2} \left[xz \right]_0^{x^2+2y^2} dy \, dx$$

$$= \int_{-3}^0 \int_{-x}^{-x+2} (x^3 + 2xy^2) dy \, dx$$

$$= \int_{-3}^0 \left[x^3 y + \frac{2xy^3}{3} \right]_{-x}^{-x+2} dx$$

$$= \int_{-3}^0 \left(x^3(-x+2) + \frac{2x(-x+2)^3}{3} - x^3(-x) - \frac{2x(-x)^3}{3} \right) dx$$

$$= \dots = \int_{-3}^0 \left(6x^3 - 8x^2 + \frac{16}{3}x \right) dx$$

$$= \dots = -\frac{435}{2}$$

Eksempel Hva er volumet over kjeglen $z = \sqrt{x^2 + y^2}$ og under $z = 2$? 2

Løsning Skjæring mellom planet $z = 2$ og kjeglen er $\sqrt{x^2 + y^2} = 2 \Leftrightarrow x^2 + y^2 = 4$, sirkel med sentrer $(0,0)$, radius 2.

Bruk sylinderkoordinater : $x = r \cos \theta$
 $y = r \sin \theta$
 $z = z$



Området er nå

$$D = \{ (r, \theta, z) : 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2, r \leq z \leq 2 \}$$

Jacobi determinanten :

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r,$$

$$|J| = r.$$

$$\text{Vol}(S) = \iiint_S 1 \, dx \, dy \, dz = \int_0^{2\pi} \int_0^2 \int_r^2 1 \cdot r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 r(2-r) \, dr \, d\theta$$

$$= 2\pi \int_0^2 (2r - r^2) \, dr$$

$$= 2\pi \left[r^2 - \frac{r^3}{3} \right]_0^2 = 2\pi \cdot \frac{4}{3} = \frac{8\pi}{3}.$$

Eksempel
Løsning

Hva er volumet til en kule med radius a .
Bruk kulekoordinater 3.

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

I kulen er koordinatene slik:

$$0 \leq \rho \leq a, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi$$

Jacobi-determinanten er da

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix}$$

$$= \rho^2 \sin \phi, \quad \Rightarrow |J| = \rho^2 \sin \phi,$$

$$\text{Vol}(S) = \iiint_S |dx dy dz| = \iiint_D 1 \cdot \left| \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} \right| d\rho d\phi d\theta$$

der

$$D = \{(\rho, \phi, \theta) : 0 \leq \rho \leq a, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}.$$

$$V(S) = \int_0^{2\pi} \int_0^\pi \int_0^a 1 \cdot \rho^2 \sin \phi d\rho d\phi d\theta$$

$$= \left(\int_0^{2\pi} 1 \cdot d\theta \right) \left(\int_0^\pi \sin \phi d\phi \right) \left(\int_0^a \rho^2 d\rho \right)$$

$$= 2\pi \left[-\cos \phi \right]_0^\pi \left[\frac{\rho^3}{3} \right]_0^a$$

$$= 2\pi \cdot 2 \cdot \frac{a^3}{3}$$

$$= \frac{4\pi a^3}{3}$$

Skifte av variable

2 variable

Teorem.

La $U \subset \mathbb{R}^2$ være åpen og begrenset.

La $T: U \rightarrow \mathbb{R}^2$ være injektiv og la

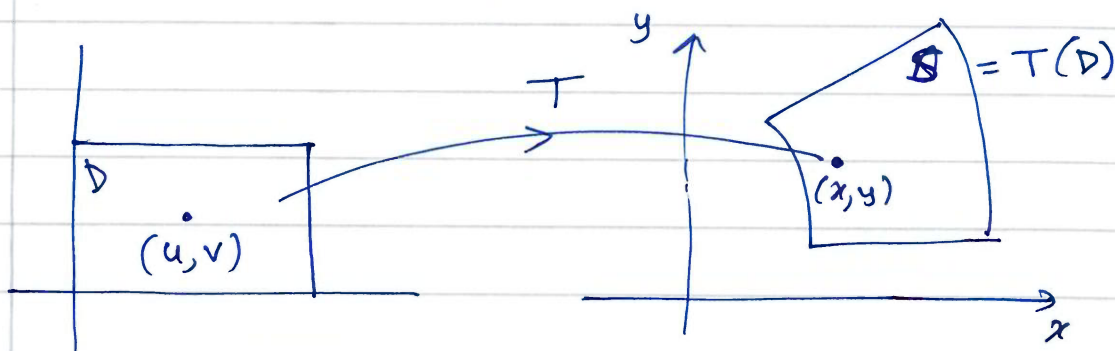
$D \subset U$ være lukket og Jordan-målbar.

Så er

$$\iint_{T(D)} f(x,y) dx dy = \iint_D f(T(u,v)) |\det T'(u,v)| du dv$$

$$= \iint_D f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$

for kontinuerlig f .



Her er $\frac{\partial(x,y)}{\partial(u,v)}$ Jacobi-determinanten

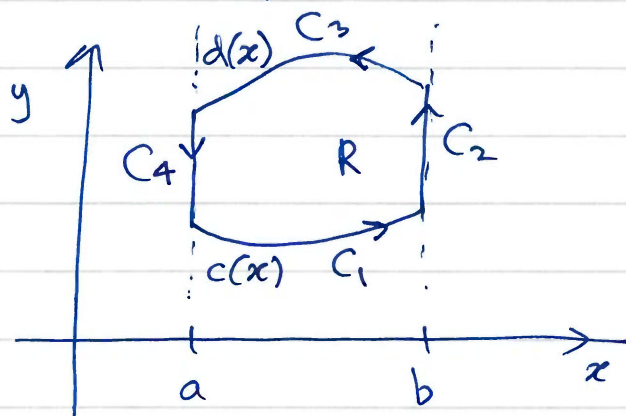
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Poenget: hvis S er et vanskelig område å integrere over, finner et enere område D og enavbildning $T: D \rightarrow S$.

Seksj. 6.5. Bevis av Green's teorem for et område som er både type-I og type-II.

kan vise at
$$\int_C P dx = - \iint_R \frac{\partial P}{\partial y} dx dy$$

For et type-I område R .



Vi parametriserer :

$$C_1 : \vec{r}_1(t) = (t, c(t)) , a \leq t \leq b$$

$$C_2 : \vec{r}_2(t) = (b, t) , c(b) \leq t \leq d(b)$$

$$C_3 : \vec{r}_3(t) = (t, d(t)) , a \leq t \leq b \quad (\text{motsatt vei})$$

$$C_4 : \vec{r}_4(t) = (a, t) , c(a) \leq t \leq d(a) \quad (\text{motsatt vei})$$

vi splitter opp :
$$\int_C P dx = \sum_{i=1}^4 \int_{C_i} P dx$$

La $\vec{r}_1(t) = (x(t), y(t)) = (t, c(t))$. Da er $x'(t) = 1$ og

$$\int_{C_1} P dx = \int_a^b P(t, c(t)) x'(t) dt = \int_a^b P(t, c(t)) dt.$$

Når $\vec{r}_2(t) = (x(t), y(t)) = (b, t)$ er $x'(t) = 0$.

Da er
$$\int_{C_2} P dx = \int_{c(b)}^{d(b)} P(b, t) \cdot 0 \cdot dt = 0.$$

$$\int_{C_3} P dx = - \int_a^b P(t, d(t)) dt, \quad \int_{C_4} P dx = 0.$$

$$\begin{aligned} \text{Da er } \int_C P dx &= \int_a^b (P(t, c(t)) - P(t, d(t))) dt \\ &= - \int_a^b \int_{c(t)}^{d(t)} \frac{\partial P}{\partial y}(t, y) dy dt \\ &= - \iint_R \frac{\partial P}{\partial y} dy dx. \end{aligned}$$

Hvis området R er også type-II, er

$$\int_C Q dy = \iint_R \frac{\partial Q}{\partial x} dx dy \quad \left(\begin{array}{l} \text{positiv} \\ \text{for tegn} \\ \text{denne gangen} \end{array} \right)$$

Dermed er

$$\int_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$