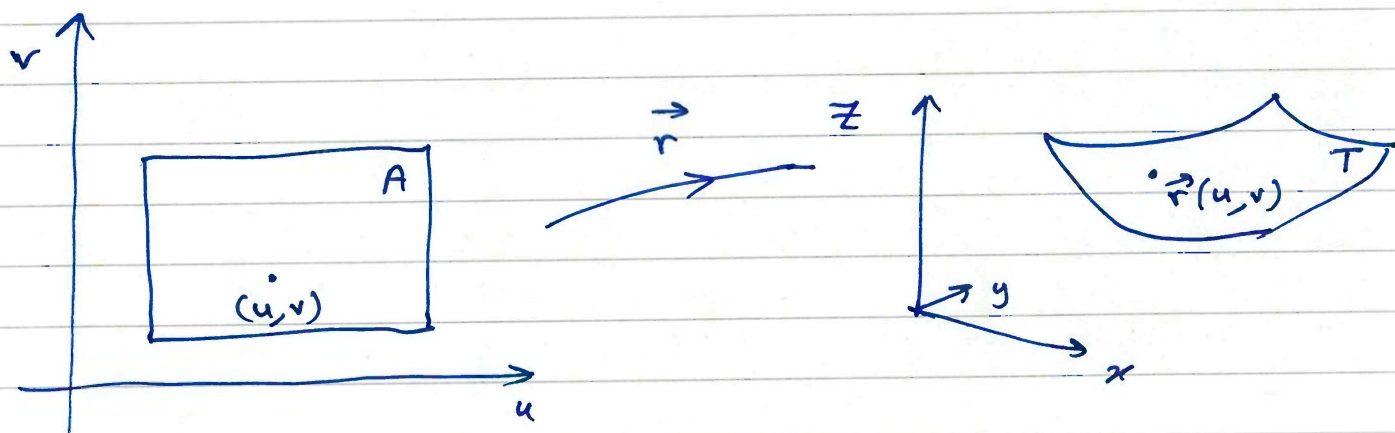


Flateintegraler av vektorfelt

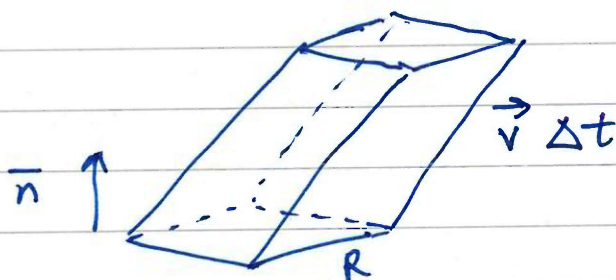
Anta at $\vec{r}: A \rightarrow \mathbb{R}^3$ er en parametrisert flate T ,
 A er lukket, begrenset, Jordan-målbart, $A \subseteq \mathbb{R}^2$.
Hvis $\vec{F}$ er et kontinuerlig vektorfelt på T
er flateintegralet til $\vec{F}$,

$$\int_T \vec{F} \cdot \vec{n} \, ds = \iint_A \vec{F}(\vec{r}(u,v)) \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right)(u,v) \, du \, dv.$$



Vi kaller $\int_T \vec{F} \cdot \vec{n} \, ds$ fluksen til $\vec{F}$ gjennom T .

Hvorfor? Anta vi har en væske eller gass
som beveger seg med hastighet $\vec{v}$.
La R være et flateelement. Volumet til
røret som strømmer gjennom R i tidsrommet
 Δt er $\vec{v} \cdot \vec{n} R \Delta t$

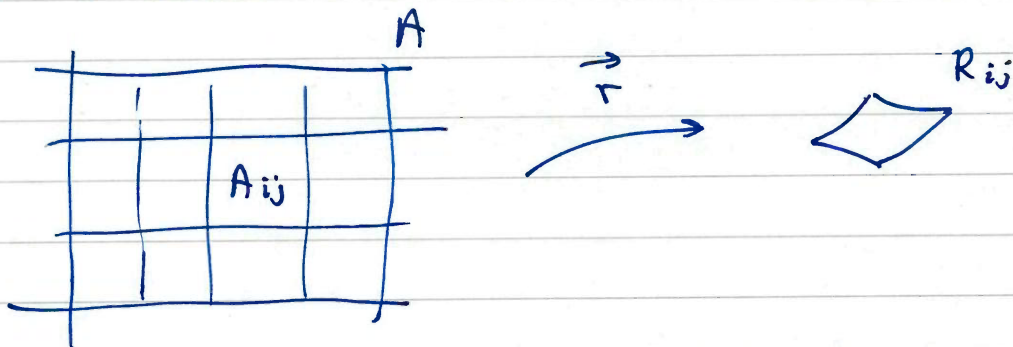


Anta at væsken har tetthet ρ . Da er massen som strømmer gjennom R per tidsenhet

$$\rho \vec{v} \cdot \vec{n} R = \vec{F} \cdot \vec{n} R,$$

der $\vec{F} = \rho \vec{v}$ er fluksen, produktet av tetthet og hastighet.

Del A i ruter



R_{ij} (arealet til R_{ij}) $\approx$

$$\begin{aligned} & |(\vec{r}(u_i, v_{j-1}) - \vec{r}(u_{i-1}, v_{j-1})) \times (\vec{r}(u_{i-1}, v_j) - \vec{r}(u_{i-1}, v_{j-1}))| \\ & \approx \left| \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right)(u_{i-1}, v_{j-1}) \right| (u_i - u_{i-1})(v_j - v_{j-1}) \end{aligned}$$

$$\text{og } (\vec{F} \cdot \vec{n})(u_{i-1}, v_{j-1}) R_{ij} \approx \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right)(u_{i-1}, v_{j-1}).$$

$$\text{Da er } \sum_{ij} (\vec{F} \cdot \vec{n}) R_{ij} \approx \iint_A F(\vec{r}(u, v)) \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right)(u, v) du dv$$

Eksempel Regn flateintegralet $\int_T \vec{F} \cdot \vec{n} dS$ der T er gitt ved $\vec{r}(u, v) = (u+v)\vec{i} + (u-v)\vec{j} + uv\vec{k}$ og

$$F(x, y, z) = 2\vec{i} + x\vec{j} + yz\vec{k}.$$

Løsning: $\frac{\partial \vec{r}}{\partial u} = \vec{i} + \vec{j} + v\vec{k}, \quad \frac{\partial \vec{r}}{\partial v} = \vec{i} - \vec{j} + u\vec{k},$
 $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = (u+v)\vec{i} - (u-v)\vec{j} - 2\vec{k}.$

$$F(\vec{r}(u,v)) \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) = 2(u+v) - (u+v)(u-v) + (u-v)uv(-2) \\ = 2u + 2v - u^2 + v^2 - 2u^2v + 2uv^2.$$

$$\int_T \vec{F} \cdot \vec{n} \, dS = \int_0^1 \left[\int_0^1 2u + 2v - u^2 + v^2 - 2u^2v + 2uv^2 \, du \right] dv \\ = \dots = 2. \\ \text{regning}$$

Det fundamentale vektorproduktet

er $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$.

Hvis $\vec{r}(u,v) = X(u,v)\vec{i} + Y(u,v)\vec{j} + Z(u,v)\vec{k}$

får vi

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} & \frac{\partial Z}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} & \frac{\partial Z}{\partial v} \end{vmatrix} = \frac{\partial(Y,Z)}{\partial(u,v)}\vec{i} + \frac{\partial(Z,X)}{\partial(u,v)}\vec{j} + \frac{\partial(X,Y)}{\partial(u,v)}\vec{k},$$

der $\frac{\partial(Y,Z)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \\ \frac{\partial Z}{\partial u} & \frac{\partial Z}{\partial v} \end{vmatrix}$ etc., Jacobi-determinanter.

Skriv F på formen $\rightarrow$

$$F(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}.$$

Da får vi

$$\int_T \vec{F} \cdot \vec{n} \, dS = \iint_A \left(P(\vec{r}(u,v)) \frac{\partial(Y,Z)}{\partial(u,v)} + Q(\vec{r}(u,v)) \frac{\partial(Z,X)}{\partial(u,v)} + R(\vec{r}(u,v)) \frac{\partial(X,Y)}{\partial(u,v)} \right) du \, dv.$$

kan bruke kortere notasjon :

4.

$$\int_T \vec{F} \cdot \vec{n} dS = \int_T (P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy),$$

$$\text{der } dy \wedge dz = \frac{\partial(y, z)}{\partial(u, v)} du dv,$$

$$dz \wedge dx = \frac{\partial(z, x)}{\partial(u, v)} du dv,$$

$$dx \wedge dy = \frac{\partial(x, y)}{\partial(u, v)} du dv.$$

Kommentarer

1. Alternativ regning : la $f = \vec{F} \cdot \vec{n}$ og beregn flateintegralet til skalarfeltet f .

2. $\int_T \vec{F} \cdot \vec{n} dS$ er uavhengig av parametriseringen $\vec{r}(u, v)$ hvis T er orienterbar,

d.v.s. T har en forside og bakside

(T er ikke et möbius bånd).

$\int_T \vec{F} \cdot \vec{n} dS$ har motsatt fortegn hvis $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ peker i motsatt normalretning.

13 Divergens og curl

La $\vec{F}$ være et vektorfelt

$$\vec{F}(x, y, z) = P(x, y, z) \vec{i} + Q(x, y, z) \vec{j} + R(x, y, z) \vec{k}$$

Vi definerer divergensen til $\vec{F}$,

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$\operatorname{div} \vec{F}$ er et skalarfelt.

Eksempel Hvis $F(x, y, z) = x^2 y z \vec{i} + x y^2 z \vec{j} + x e^{y z} \vec{k}$,

$$\begin{aligned} \text{er } \operatorname{div} \vec{F}(x, y, z) &= 2xy z + 2xy z + xy e^{yz} \\ &= 4xy z + xy e^{yz}. \end{aligned}$$

Vi definerer curlen til $\vec{F}$ ved

$$\operatorname{curl} \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}.$$

$\operatorname{curl} \vec{F}$ er et vektorfelt.

Lettere å huske formelen som en determinant,

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x, y, z) & Q(x, y, z) & R(x, y, z) \end{vmatrix}$$

Eksempel Finn $\text{curl } \vec{F}$ når

$$F(x, y, z) = (x^2 y + z) \vec{i} + z \sin(xy) \vec{j} + (x + y^2) \vec{k}$$

Løsning $\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y + z & z \sin(xy) & x + y^2 \end{vmatrix}$

$$= (2y - \sin(xy)) \vec{i} + 0 \vec{j} + (zy \cos(xy) - x^2) \vec{k}.$$

Alternativ notasjon: definer differensialoperatoren

$$\vec{\nabla} = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}$$

Da er

$$\begin{aligned} \text{div } \vec{F} &= \vec{\nabla} \cdot \vec{F} = \left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right) \cdot (P \vec{i} + Q \vec{j} + R \vec{k}) \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \end{aligned}$$

$$\begin{aligned} \text{curl } \vec{F} &= \vec{\nabla} \times \vec{F} = \left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right) \times (P \vec{i} + Q \vec{j} + R \vec{k}) \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}. \end{aligned}$$

Kombinasjoner av operatorer

Vi definerer Laplace-operatoren ved

$$\Delta f = \operatorname{div}(\nabla f).$$

$$\text{Vi får } \Delta f = \operatorname{div}\left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

curl av gradienten

$$\begin{aligned}\operatorname{curl}(\nabla f) &= \operatorname{curl}\left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k\right) \\ &= \left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right) i + \left(\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)\right) j \\ &\quad + \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) k = 0.\end{aligned}$$

div av curl

$$\begin{aligned}\operatorname{div}(\operatorname{curl} \vec{F}) &= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\ &= \left(\frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z}\right) + \left(\frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x}\right) + \left(\frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y}\right) = 0.\end{aligned}$$