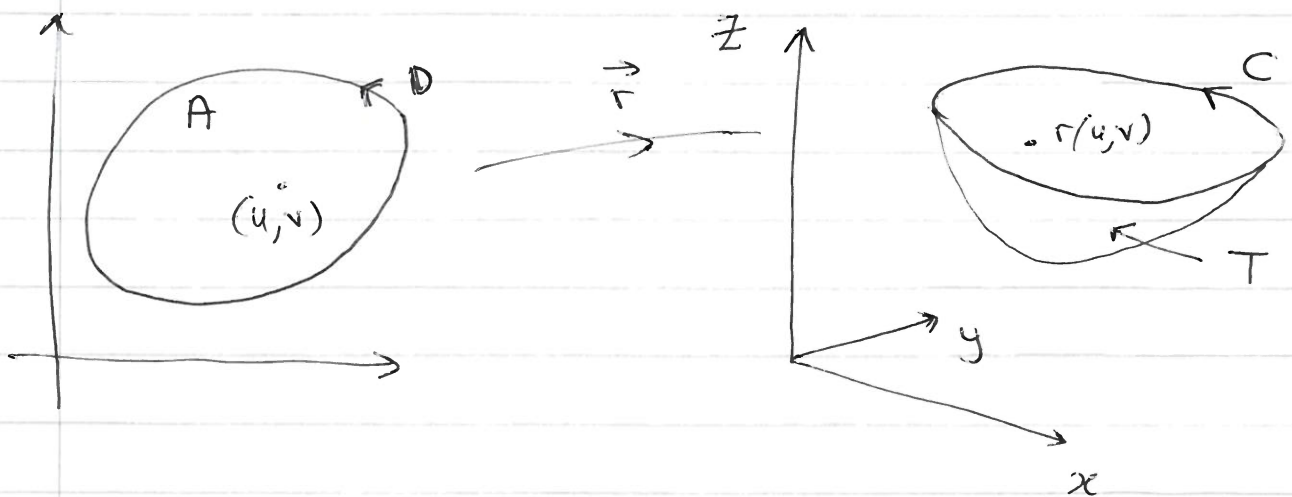


Stokes' teorem

La D være en positivt orientert kurve i (u,v) -planet, som avgrenser et område A .
La T være den parametriserte flaten gitt ved $\vec{r}: A \rightarrow \mathbb{R}^3$, og la $C = \vec{r}(D)$.
Da er

$$\int_T \text{curl } \vec{F} \cdot \vec{n} \, dS = \int_C \vec{F} \cdot d\vec{r}$$

For et vektorfelt $\vec{F}$ med kontinuerlige part. deriv. i T .



Bevis $\int_T \text{curl} \vec{F} \cdot \vec{n} \, ds$

$$= \int_T \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \int_T \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \int_T \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy,$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + \int_C Q dy + \int_C R dz.$$

Nok å vise at

$$\textcircled{1} \quad \int_C P dx = \int_T \frac{\partial P}{\partial z} dz \wedge dx - \int_T \frac{\partial P}{\partial y} dx \wedge dy$$

$$\textcircled{2} \quad \int_C Q dy = - \int_T \frac{\partial Q}{\partial z} dy \wedge dz + \int_T \frac{\partial Q}{\partial x} dx \wedge dy$$

$$\textcircled{3} \quad \int_C R dz = \int_T \frac{\partial R}{\partial y} dy \wedge dz - \int_T \frac{\partial R}{\partial x} dz \wedge dx.$$

Skål bare vise $\textcircled{1}$.

$$\text{La } \vec{r}(u,v) = X(u,v) \vec{i} + Y(u,v) \vec{j} + Z(u,v) \vec{k}$$

og la kurven D i (u,v) -planet ha parametrisering

$$s(t) = u(t) \vec{i} + v(t) \vec{j}, \quad t \in [a,b]$$

Da har kurven C parametriseringen

$$\begin{aligned} \vec{r}(t) &= x(t) \vec{i} + y(t) \vec{j} + z(t) \vec{k} \\ &= X(u(t), v(t)) \vec{i} + Y(u(t), v(t)) \vec{j} + Z(u(t), v(t)) \vec{k} \end{aligned}$$

$$\text{Kjerneregelen: } x'(t) = \frac{\partial X}{\partial u} u'(t) + \frac{\partial X}{\partial v} v'(t),$$

og da er

$$\begin{aligned} \int_C P dx &= \int_a^b P(\vec{r}(t)) x'(t) dt \\ &= \int_a^b p(u(t), v(t)) x'(t) dt \end{aligned}$$

$$= \int_D P \frac{\partial X}{\partial u} du + P \frac{\partial X}{\partial v} dv,$$

$$\text{der } p(u,v) := P(X(u,v), Y(u,v), Z(u,v)).$$

Nå bruker vi Greens teorem i området A :

$$\int_D \left(P \frac{\partial X}{\partial u} \right) du + \left(P \frac{\partial X}{\partial v} \right) dv$$

$$= \iint_A \frac{\partial}{\partial u} \left(P \frac{\partial X}{\partial v} \right) - \frac{\partial}{\partial v} \left(P \frac{\partial X}{\partial u} \right) du dv$$

$$= \iint_A \frac{\partial P}{\partial u} \frac{\partial X}{\partial v} + P \frac{\partial^2 X}{\partial u \partial v} - \left(\frac{\partial P}{\partial v} \frac{\partial X}{\partial u} + P \frac{\partial^2 X}{\partial v \partial u} \right) du dv$$

$$= \iint_A \frac{\partial P}{\partial u} \frac{\partial X}{\partial v} - \frac{\partial P}{\partial v} \frac{\partial X}{\partial u} du dv.$$

Kjernerregelen:

$$\frac{\partial P}{\partial u} = \frac{\partial P}{\partial x} \frac{\partial X}{\partial u} + \frac{\partial P}{\partial y} \frac{\partial Y}{\partial u} + \frac{\partial P}{\partial z} \frac{\partial Z}{\partial u}$$

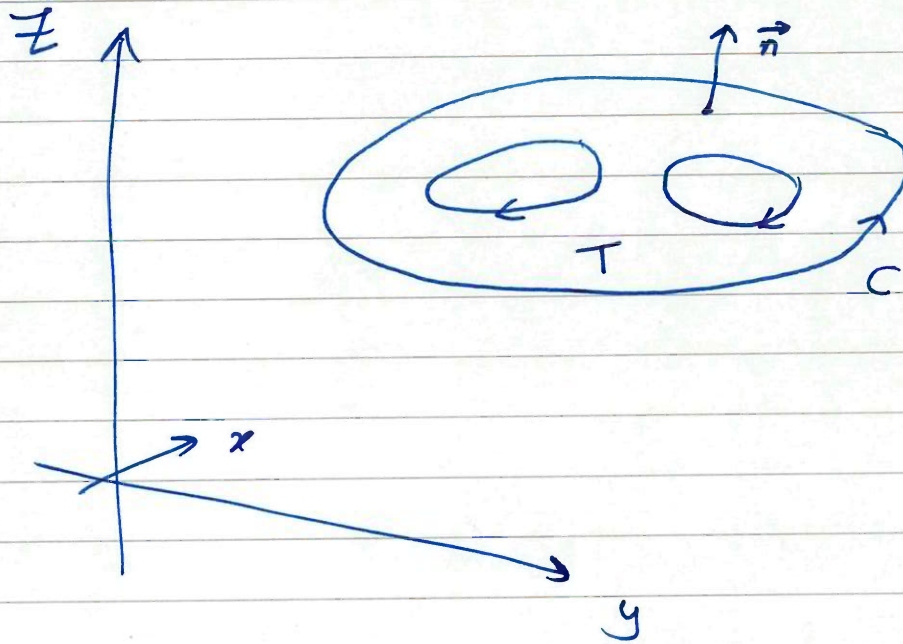
$$\frac{\partial P}{\partial v} = \frac{\partial P}{\partial x} \frac{\partial X}{\partial v} + \frac{\partial P}{\partial y} \frac{\partial Y}{\partial v} + \frac{\partial P}{\partial z} \frac{\partial Z}{\partial v}$$

$\Rightarrow$

$$\int_C P dx = \iint_A \frac{\partial P}{\partial z} \left(\frac{\partial(Z, X)}{\partial(u, v)} \right) - \frac{\partial P}{\partial y} \left(\frac{\partial(X, Y)}{\partial(u, v)} \right) du dv$$

$$= \int_T \frac{\partial P}{\partial z} dz \wedge dx - \int_T \frac{\partial P}{\partial y} dx \wedge dy.$$

Overflaten T kan ha hull :



Regelen: Flaten T skal alltid ligge på
venstre hånd når du spaserer rundt
randen C med hodet i den retningen
enhetsnormalen $\vec{n}$ peker.

Seksjon 5.1. Topologi i $\mathbb{R}^m$

$$\begin{aligned} B(\vec{a}, r) &= \left\{ \vec{x} \in \mathbb{R}^m : |\vec{x} - \vec{a}| < r \right\} \text{ \u00f8pen kule} \\ \overline{B}(\vec{a}, r) &= \left\{ \vec{x} \in \mathbb{R}^m : |\vec{x} - \vec{a}| \leq r \right\} \text{ lukket kule} \end{aligned}$$

$\angle a$ $A \subseteq \mathbb{R}^m$ og la $\vec{a} \in \mathbb{R}^m$. $\vec{a}$ kalles

indrepunkt for A hvis det finnes en kule $B(\vec{a}, r)$ inneholdt i A

randpunkt for A hvis enhver $B(\vec{a}, r)$ inneholder punkter b\u00e5de i A og ikke i A ,

ytrepunkt for A hvis det finnes $B(\vec{a}, r)$ som ikke inneholder noen punkter i A .

$A \subseteq \mathbb{R}^m$ kalles

lukket hvis den inneholder alle sine randpunkter,
\u00f8pen " " ikke inneholder noen randpunkter.

Definisjon Vi sier at f\u00f8lgen $\{\vec{x}_n\}$ konvergerer

mot $\vec{a} \in \mathbb{R}^m$ hvis for enhver $\varepsilon > 0$ finnes det $N > 0$ slik at $|\vec{x}_n - \vec{a}| < \varepsilon$ for alle $n \geq N$.

Vi skriver

$$\lim_{n \rightarrow \infty} \vec{x}_n = \vec{a} \quad \text{eller} \quad \vec{x}_n \rightarrow \vec{a}.$$

Setning 5.1.6. Anta at $\{\vec{x}_n\}$ er en f\u00f8lge i $\mathbb{R}^m$ med komponenter

$$\vec{x}_n = (x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)})$$

og la $\vec{x} = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$. Da er

$$\lim_{n \rightarrow \infty} \vec{x}_n = \vec{x}$$

hvis og bare hvis

$$\lim_{n \rightarrow \infty} x_i^{(n)} = x_i \quad \text{for alle } i = 1, 2, \dots, m.$$

Beris Anta først at $\lim_{n \rightarrow \infty} \vec{x}_n = \vec{x}$ og

(la $\varepsilon > 0$. Da finnes det $N > 0$ slik at
 $|\vec{x}_n - \vec{x}| < \varepsilon$ for alle $n \geq N$. For hver $i=1, \dots, m$ er

$$\begin{aligned} |x_i^{(n)} - x_i| &= \sqrt{(x_i^{(n)} - x_i)^2} \\ &\leq \sqrt{(x_1^{(n)} - x_1)^2 + \dots + (x_m^{(n)} - x_m)^2} \\ &= |\vec{x}_n - \vec{x}| \end{aligned}$$

(og da er $|x_i^{(n)} - x_i| < \varepsilon$ for alle $n \geq N$.

Anta nå at $\lim_{n \rightarrow \infty} x_i^{(n)} = x_i$ for alle $i=1, 2, \dots, m$

og la $\varepsilon > 0$. Siden $\lim_{n \rightarrow \infty} x_i^{(n)} = x_i$ finnes det

$N_i > 0$ slik at $|x_i^{(n)} - x_i| < \frac{\varepsilon}{\sqrt{m}}$ for $n \geq N_i$.

La $N = \max \{N_1, N_2, \dots, N_m\}$. For $n \geq N$ er da

$$\begin{aligned} |\vec{x}_n - \vec{x}| &= \sqrt{(x_1^{(n)} - x_1)^2 + \dots + (x_m^{(n)} - x_m)^2} \\ &\leq \sqrt{\left(\frac{\varepsilon}{\sqrt{m}}\right)^2 + \dots + \left(\frac{\varepsilon}{\sqrt{m}}\right)^2} \\ &= \sqrt{m \frac{\varepsilon^2}{m}} = \sqrt{\varepsilon^2} = \varepsilon. \end{aligned}$$

□

Eksempel

Finne grenseverdien

$$\lim_{n \rightarrow \infty} \begin{bmatrix} \frac{n^2}{n^2+1} \\ n \sin\left(\frac{1}{n}\right) \\ \left(1 + \frac{2}{n}\right)^n \end{bmatrix}$$

Løsning Det er nok å finne grensene til komponentene.

$$\lim_{n \rightarrow \infty} \frac{n^2}{n^2+1} = \lim_{n \rightarrow \infty} \frac{1}{1 + 1/n^2} = \frac{1}{1} = 1$$

$$\lim_{n \rightarrow \infty} n \sin\left(\frac{1}{n}\right) = \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \text{L'Hopital}$$

$$= \lim_{n \rightarrow \infty} \frac{\cos\left(\frac{1}{n}\right) \left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \cos\left(\frac{1}{n}\right) = 1$$

$$\left(1 + \frac{2}{n}\right)^n = e^{n \log(1+2/n)}$$

$$\lim_{n \rightarrow \infty} n \log\left(1 + \frac{2}{n}\right) = \lim_{n \rightarrow \infty} \frac{\log\left(1 + \frac{2}{n}\right)}{\frac{1}{n}} = \text{L'Hopital}$$

$$= \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{1+2/n}\right) \left(-\frac{2}{n^2}\right)}{-\left(\frac{1}{n^2}\right)} = 2$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n = e^2$$

Svaret er $\begin{bmatrix} 1 \\ 1 \\ e^2 \end{bmatrix}$.