

14/09/2026

Avsnitt 5.6 Dynamiske systemer.

Formål: å modellere et system (biologisk, fysisk, finansiell ...) som utvikler seg i tid.

Tiden registreres her ved $k=0, k=1, k=2, \dots$ (positive heltall) (Diskretisering). Her k kan stå for: dag, måned, år.

Matematisk ser på: differensligning: $\boxed{x_{k+1} = A x_k}$, $k=0, 1, 2, 3, \dots$

x_0 : tilstand ved starten, $x_0 \in \mathbb{R}^n$ (for eksempel)

x_1 : tilstand ved tidsenhet $k=1$, $x_1 \in \mathbb{R}^n$

$\vdots$
 x_k : ——— ——— $k > 1$, $x_k \in \mathbb{R}^n$

A : $n \times n$ matrise og $\boxed{(1) \ x_{k+1} = A x_k, \ k=0, 1, 2, \dots}$ *dyn. syst.*

beskriver hvordan systemet utvikler seg fra k til $k+1$.

Vi skiller mellom ulike oppførsel av systemet, utifra om $|\lambda|$ er 1, < 1 , > 1 , for λ egenverdi til A .

Eksempel ① $A = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$, $x_0 = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}$.

Beskriv oppførsel av dyn. syst. (1).

$$\det(A - \lambda I) = 0 \Leftrightarrow \det \begin{bmatrix} 0.6 - \lambda & 0.3 \\ 0.4 & 0.7 - \lambda \end{bmatrix} = 0$$

$$\Leftrightarrow (0.6 - \lambda)(0.7 - \lambda) - 0.3 \cdot 0.4 = 0$$

$$\Leftrightarrow \lambda^2 - 1.3\lambda + 0.3 = 0. \text{ Løsninger } \lambda_1 = 1 \text{ og } \lambda_2 = 0.3$$

Tilfellet $\boxed{|\lambda_1| = 1 > |\lambda_2|}$

Egenvektor hørende til $\lambda_1 = 1$: $\begin{bmatrix} -0.4 & 0.3 & : & 0 \\ 0.4 & -0.3 & : & 0 \end{bmatrix} \sim \begin{bmatrix} -0.4 & 0.3 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{3}{4} & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix}$

$x_1 = \frac{3}{4} x_2 \Rightarrow$ velg $\boxed{v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}}$ egenvektor til $\lambda_1 = 1$. $\uparrow$ fri var. $\uparrow$ fri

Egenvektor til $\lambda_2 = 0.3$ $\begin{bmatrix} 0.3 & 0.3 & : & 0 \\ 0.4 & 0.4 & : & 0 \end{bmatrix} \sim \begin{bmatrix} 0.3 & 0.3 \\ 0.4 & 0.4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix}$ løsninger $x_1 + x_2 = 0$
Velg $\boxed{v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$ for $\lambda_2 = 0.3$ $\uparrow$ x_2 fri

14/09/2026
(2)

$$\underline{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \text{ til } \lambda_1 = 1$$

$$\underline{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ til } \lambda_2 = 0.3.$$

$$(1) \quad \underline{x}_{h+1} = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \cdot \underline{x}_h ; \quad \underline{x}_0 = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}.$$

Finne formel for $\underline{x}_h$ ved hjælp af $\underline{x}_0$.

Vet: $\{\underline{v}_1, \underline{v}_2\}$ er en basis for $\mathbb{R}^2$.

Skriv: $\underline{x}_0 = c_1 \underline{v}_1 + c_2 \underline{v}_2$ for unikke skalarer c_1, c_2 .

$$\begin{aligned} (1) \Rightarrow \underline{x}_1 &= A \cdot \underline{x}_0 = A \cdot (c_1 \underline{v}_1 + c_2 \underline{v}_2) \\ &= c_1 (A \underline{v}_1) + c_2 (A \underline{v}_2) \quad \text{af lineæritet} \\ &= c_1 \lambda_1 \underline{v}_1 + c_2 \lambda_2 \underline{v}_2. \end{aligned}$$

$$\begin{aligned} \underline{x}_2 &= A \cdot \underline{x}_1 = A(c_1 \lambda_1 \underline{v}_1 + c_2 \lambda_2 \underline{v}_2) \\ &= c_1 \lambda_1 (A \underline{v}_1) + c_2 \lambda_2 (A \underline{v}_2) \\ &= c_1 \lambda_1^2 \underline{v}_1 + c_2 \lambda_2^2 \underline{v}_2. \end{aligned}$$

Ved induktion: $\underline{x}_h = c_1 \lambda_1^h \underline{v}_1 + c_2 \lambda_2^h \underline{v}_2$, for alle $h \geq 0$. (3)

$\underbrace{\quad}_{\text{skalar}} \underbrace{\quad}_{\text{vektor}} \quad \underbrace{\quad}_{\text{skalar}} \underbrace{\quad}_{\text{vektor}}$

Når vi vælger en $\underline{x}_0$ kan vi finde tilhørende c_1, c_2 :

$$\text{Løser } [\underline{v}_1 \quad \underline{v}_2] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underline{x}_0 = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\underline{v}_1 \quad \underline{v}_2]^{-1} \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}. \quad \text{Finne } \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}^{-1} = \frac{1}{-7} \begin{bmatrix} -1 & -1 \\ -4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0.075 \\ 0.037 \end{bmatrix}.$$

Her er $\lambda_1 = 1$ så $\lambda_1^h = 1$ og $\underline{x}_h = c_1 \underline{v}_1 + c_2 (0.3)^h \underline{v}_2$, $h \geq 0$.

Når $h \rightarrow \infty$, er $\lim_{h \rightarrow \infty} (0.3)^h = 0$. (Kalkulus: for $a \in \mathbb{R}$, $|a| < 1$ er $\lim_{h \rightarrow \infty} a^h = 0$.)

altså, $\lim_{h \rightarrow \infty} \underline{x}_h = c_1 \underline{v}_1 = \begin{bmatrix} 0.075 \cdot 3 \\ 0.037 \cdot 4 \end{bmatrix}$ konstant.

← Kaller likvektstilstand (for start $\underline{x}_0$).

$$|\lambda_1| = 1, |\lambda_2| < 1: \quad \underline{x}_{h+1} \approx c_1 \lambda_1^{h+1} \underline{v}_1 = \lambda_1 \underline{x}_h, \text{ og } \underline{x}_h \approx c_1 \underline{v}_1.$$

□

Generelt: A $n \times n$ matrise

14/09/2026

③

med egenverdier: $\lambda_1, \lambda_2, \dots, \lambda_n$, ordnet med:

$$|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|.$$

Ser etter, om $|\lambda_i| > 1$ eller $|\lambda_j| < 1$ blandt λ 'er.

Gitt $\underline{x}_0 \in \mathbb{R}^n$, mengden av punkter $\underline{x}_0, \underline{x}_1 = A \underline{x}_0, \underline{x}_2 = A \underline{x}_1, \dots, \underline{x}_k, \dots$
kalles banen til dyn. syst. gitt ved A .

Ulike valg av $\underline{x}_0 \rightsquigarrow$ ulike baner.

Eksempel ② (Eks. 1 i avsnitt 5.6) Her er tilfellet $|\lambda_1| > 1 > |\lambda_2|$

$$A = \begin{bmatrix} 0.5 & 0.4 \\ 0.104 & 1.1 \end{bmatrix}. \text{ Det er oppgitt, at } \lambda_1 = 1.02 \text{ og } \lambda_2 = 0.58,$$

samt, at $\underline{v}_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix}$ er en egenvektor til λ_1

$$\text{og } \underline{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \text{ er en egenvektor til } \lambda_2.$$

Ønsker: se på dyn. syst. $\underline{x}_{k+1} = A \underline{x}_k, k=0, 1, 2, \dots$. Beskriv dette når $k \rightarrow \infty$.

Har: $\{\underline{v}_1, \underline{v}_2\}$ basis for $\mathbb{R}^2$.

Gitt $\underline{x}_0 \in \mathbb{R}^2$, finnes det unike $c_1, c_2 \in \mathbb{R}$:

$$\underline{x}_0 = c_1 \underline{v}_1 + c_2 \underline{v}_2$$

Som i eks. ① er $\underline{x}_k = c_1 \lambda_1^k \underline{v}_1 + c_2 \lambda_2^k \underline{v}_2, k \geq 1$.

$$\underline{x}_k = c_1 (1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} + c_2 (0.58)^k \begin{bmatrix} 5 \\ 1 \end{bmatrix}, k \geq 1.$$

$$\lim_{k \rightarrow \infty} (1.02)^k = \infty$$

$$\lim_{k \rightarrow \infty} (0.58)^k = 0$$

Dersom $c_1 > 0$ vil $\underline{x}_k \approx c_1 (1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix}$ når k vokser.

Merk: $\underline{x}_{k+1} \approx c_1 (1.02)^{k+1} \begin{bmatrix} 10 \\ 13 \end{bmatrix} = (1.02) \cdot \underline{x}_k$.

Origo $(0,0)$ er et saddelpunkt: $\leftarrow$ frastøter retninger langs $\underline{0}$ til $\underline{v}_1$
 $\rightarrow$ attraktor $\rightarrow$ langs $\underline{0}$ til $\underline{v}_2$.

14/09/2026
(4)

Eksempel (3) (Eks. 2 i 5.6)

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.64 \end{bmatrix}, \quad \lambda_1 = 0.8, \quad \lambda_2 = 0.64.$$

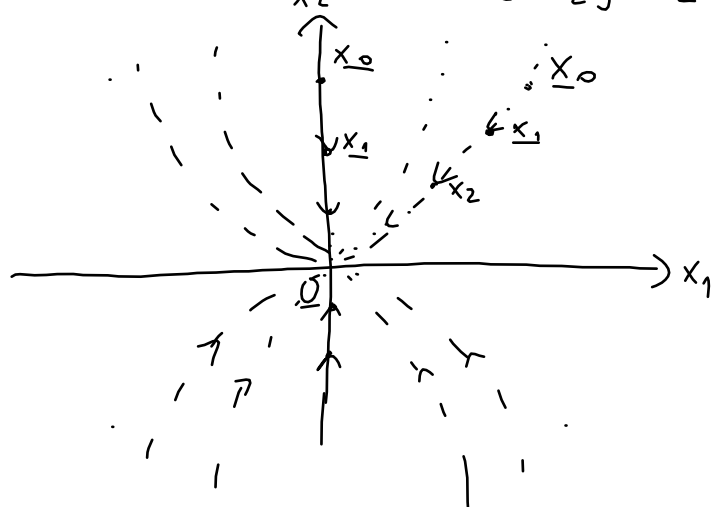
Tilfelle: $1 > |\lambda_1| > |\lambda_2|$.
 $\downarrow$ velg egenvektor $\downarrow$ velg egenvektor
 $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Gitt $\underline{x}_0 \in \mathbb{R}^2$, $\underline{x}_0 = c_1 \underline{v}_1 + c_2 \underline{v}_2$

op $\underline{x}_k = c_1 \lambda_1^k \underline{v}_1 + c_2 \lambda_2^k \underline{v}_2, \quad k = 1, 2, \dots$

$\lim_{k \rightarrow \infty} (0.8)^k = \lim_{k \rightarrow \infty} (0.64)^k = 0$ så $\underline{x}_k \approx 0$ når $k \rightarrow \infty$.

F.eks: $\underline{x}_0 = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \rightsquigarrow \underline{x}_1 = \begin{bmatrix} c_1 \cdot \lambda_1 \\ c_2 \cdot \lambda_2 \end{bmatrix} = \begin{bmatrix} 2.4 \\ 1.92 \end{bmatrix} \rightsquigarrow \underline{x}_2 = \begin{bmatrix} c_1 \cdot \lambda_1^2 \\ c_2 \cdot \lambda_2^2 \end{bmatrix}$ osv.



For $\underline{x}_0 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $\underline{x}_1 = \begin{bmatrix} 0 \\ 3 \cdot \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.92 \end{bmatrix}$

$\underline{x}_2 = \begin{bmatrix} 0 \\ 3 \cdot \lambda_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.22 \end{bmatrix}$.

0: attraktor for banene (til hvert valg av $\underline{x}_0$)
 $1 > |\lambda_1| > |\lambda_2|$.

Eks. ④ (Eks 3 i 5.6)

(Tilfellet $|\lambda_1| > |\lambda_2| > 1$)

$$A = \begin{bmatrix} 1.44 & 0 \\ 0 & 1.2 \end{bmatrix}$$

$$\lambda_1 = 1.44, \quad \lambda_2 = 1.2$$

↓
velg $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ velg $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
egenvektorer.

For $\underline{x}_0 = C_1 \underline{v}_1 + C_2 \underline{v}_2 = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$

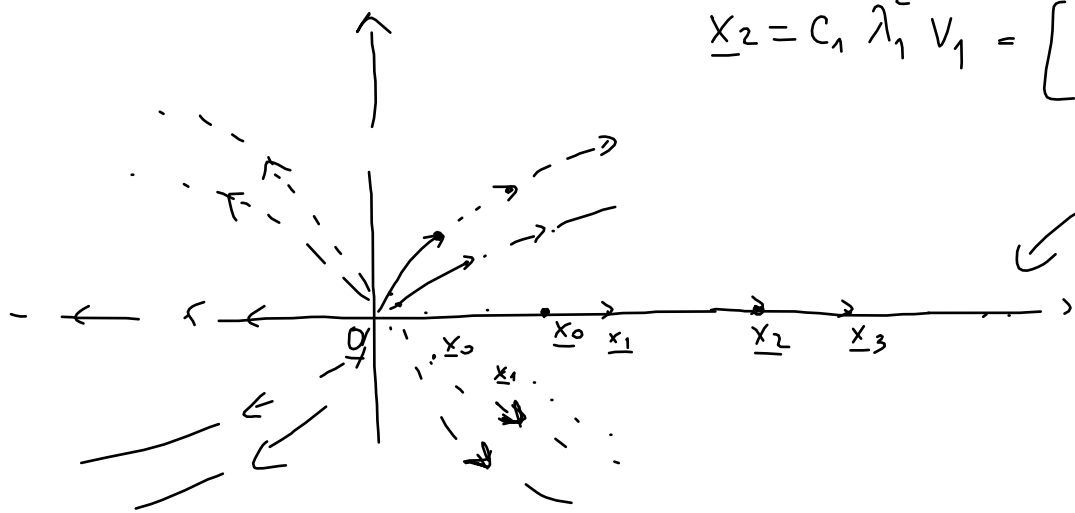
og $\underline{x}_k = C_1 \lambda_1^k \underline{v}_1 + C_2 \lambda_2^k \underline{v}_2 = C_1 \underbrace{(1.44)^k}_{\downarrow k \rightarrow \infty} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 \underbrace{(1.2)^k}_{\downarrow k \rightarrow \infty} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$

Her er origo en frastøter for banene når $k \rightarrow \infty$.

F.eks: $\underline{x}_0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ er $C_2 = 0$ og $\underline{x}_1 = C_1 \cdot \lambda_1 \underline{v}_1 = \begin{bmatrix} 3 \cdot 1.44 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.32 \\ 0 \end{bmatrix}$

$$\underline{x}_2 = C_1 \lambda_1^2 \underline{v}_1 = \begin{bmatrix} 3 \cdot (1.44)^2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6.22 \\ 0 \end{bmatrix}$$

↙ størst effekt av frastøter.



Eks. ⑤ (Eks 4)

⑥

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$\lambda_1 = 2, \lambda_2 = 0.5$$

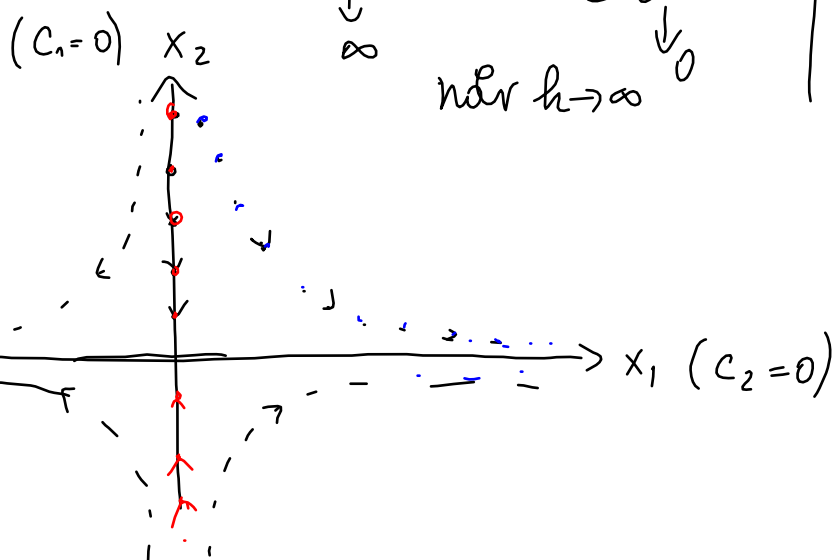
$$\begin{matrix} \uparrow & \uparrow \\ \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{matrix} \text{ egen vektorer}$$

Tilfellet $| \lambda_1 | > 1 > | \lambda_2 |$

Dyn. System: $\underline{y}_{k+1} = D \underline{y}_k$

$$\underline{y}_k = c_1 \lambda_1^k \underline{v}_1 + c_2 \lambda_2^k \underline{v}_2 \text{ for } \underline{y}_0 = c_1 \underline{v}_1 + c_2 \underline{v}_2$$

$$\underline{y}_k = c_1 \cdot 2^k \underline{v}_1 + c_2 \cdot \left(\frac{1}{2}\right)^k \underline{v}_2$$



$$\underline{y}_0 = \begin{bmatrix} c_1 \\ 0 \end{bmatrix} \Rightarrow \underline{y}_k = \begin{bmatrix} 2^k c_1 \\ 0 \end{bmatrix} \rightarrow \infty \text{ i 1ste koordinat}$$

$$\underline{y}_0 = \begin{bmatrix} 0 \\ c_2 \end{bmatrix} \Rightarrow \underline{y}_k = \begin{bmatrix} 0 \\ \left(\frac{1}{2}\right)^k c_2 \end{bmatrix} \rightarrow 0 \text{ origo } k \rightarrow \infty$$

$\underline{0}$: saddeelpunkt.

Eks:

$$A = \begin{bmatrix} 1.25 & -0.75 \\ -0.75 & 1.25 \end{bmatrix} \rightarrow \lambda_1 = 2, \lambda_2 = 0.5 \text{ (som for D)}$$

$$\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Skiftet basis: til $\{\underline{v}_1, \underline{v}_2\}$

$$\underline{x}_{k+1} = A \underline{x}_k \xrightarrow{\text{variabel skifte}} \underline{y}_{k+1} = D \underline{y}_k \text{ hvor}$$

$$A = P D P^{-1} : P \underline{x}_k = \underline{y}_k$$

□