

05/10/2026

①

Avsnitt 6.2 Ortogonalitet i $\mathbb{R}^n$ Def: $\{\underline{v}_1, \dots, \underline{v}_p\}$ i $\mathbb{R}^n$ er en ortogonal mengde dersom

$$\underline{v}_j \cdot \underline{v}_k = 0 \text{ for alle } j \neq k, j, k \in \{1, \dots, p\}.$$

(To forskjellige vektorer i mengden er parvis ortogonale!)

Merk: Ikke noen betingelser på $\underline{v}_j \cdot \underline{v}_j$, $j=1, \dots, n$.Sier: $\{\underline{v}_1, \dots, \underline{v}_p\}$ er en ortogonal basis for W underrom

dersom den er : - basis

- ortogonal.

Ser at: når $\{\underline{v}_1, \dots, \underline{v}_p\}$ er ortogonal og $\underline{v}_j \neq \underline{0}$, $j=1, \dots, p$, er $\{\underline{v}_1, \dots, \underline{v}_p\}$ en ortogonal basis for $W = \text{span}\{\underline{v}_1, \dots, \underline{v}_p\}$.Projeksjon på en vektor:Gitt $\underline{u} \in \mathbb{R}^n$, $\underline{u} \neq \underline{0}$. For $\underline{x} \in \mathbb{R}^n$, ønsker:

$$\underline{x} = \hat{\underline{x}} + \underline{z}, \quad \hat{\underline{x}} \text{ har samme retning som } \underline{u} \text{ og } \underline{z} \cdot \underline{u} = 0.$$

Her: La $\hat{\underline{x}} = \frac{\underline{x} \cdot \underline{u}}{\underline{u} \cdot \underline{u}} \underline{u}$ og klart at $\hat{\underline{x}}, \underline{u}$ har samme retning. ($\underline{z} \perp \underline{u}$)

skalare

$$\underline{z} = \underline{x} - \hat{\underline{x}}. \quad \text{Sjekk: } \underline{z} \cdot \underline{u} = (\underline{x} - \hat{\underline{x}}) \cdot \underline{u} = \underline{x} \cdot \underline{u} - \hat{\underline{x}} \cdot \underline{u}$$

 $\hat{\underline{x}}$ = projeksjonen av $\underline{x}$ på $\underline{u}$ (langs $\underline{u}$)

$$\begin{aligned} &= \underline{x} \cdot \underline{u} - \frac{\underline{x} \cdot \underline{u}}{(\underline{u} \cdot \underline{u})} (\underline{u} \cdot \underline{u}) \\ &= \underline{x} \cdot \underline{u} - \underline{x} \cdot \underline{u} = 0. \end{aligned}$$

≠ 0 skalar

Skriver også: $L = \text{span}\{\underline{u}\}$, en linje;

$$\boxed{\hat{\underline{x}} = \text{proj}_L \underline{x}.$$

Eksempel: $\underline{u} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$, $\underline{x} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$, $L = \text{span}\{\underline{u}\}$.

$$\underline{x} \cdot \underline{u} = 4 \cdot 8 + 6 \cdot 4 = 56$$

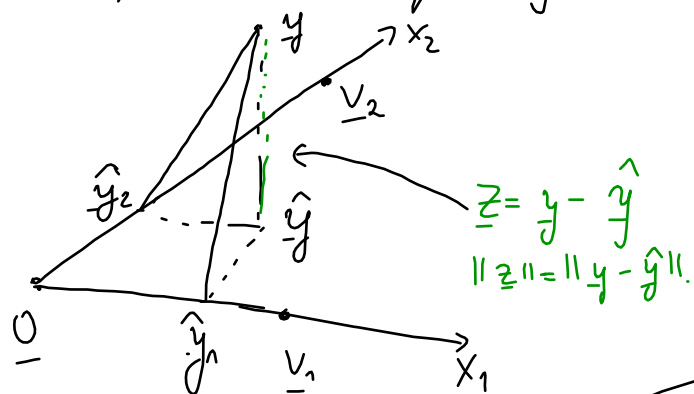
$$\underline{u} \cdot \underline{u} = 8^2 + 4^2 = 80$$

$$\hat{\underline{x}} = \frac{56}{80} \cdot \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{56}{10} \\ \frac{56}{20} \end{bmatrix} = \begin{bmatrix} 28/5 \\ 14/5 \end{bmatrix}.$$

$$\underline{z} = \underline{x} - \hat{\underline{x}} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 28/5 \\ 14/5 \end{bmatrix} = \begin{bmatrix} -8/5 \\ 16/5 \end{bmatrix}. \quad \text{Sjekk: } \underline{z} \cdot \underline{u} = 0.$$

Ortogonal projeksjon på et underrom av $\mathbb{R}^n$:

$\mathbb{R}^3$: $\underline{v}_1, \underline{v}_2 \neq 0$, gitt $\underline{y}$. $W = \text{span}\{\underline{v}_1, \underline{v}_2\}$.



Ønsker: projisere $\underline{y}$ på W

$$\underline{y} = \hat{\underline{y}} + \underline{z}$$

hvor $\hat{\underline{y}} \in W$

hvor $\underline{z} \in W^\perp$

$$\hat{\underline{y}} = \hat{\underline{y}}_1 + \hat{\underline{y}}_2$$

$$\hat{\underline{y}}_1 = \text{proj}_{L_1} \hat{\underline{y}}, \quad L_1 = \text{span}\{\underline{v}_1\}$$

$$\hat{\underline{y}}_2 = \text{proj}_{L_2} \hat{\underline{y}}, \quad L_2 = \text{span}\{\underline{v}_2\}$$

Defineres som den ortogonale projeksjon av $\underline{y}$ på W .

Viktig: Ønsker $\{\underline{v}_1, \underline{v}_2\}$ ortogonal mengde.

Generelt:

Teorem 8: (Ortogonal dekomposisjon)

Gitt W underrom av $\mathbb{R}^n$ (ikke-trivielt). Enhver $\underline{y} \in \mathbb{R}^n$ skrives entydig på form

$$(1) \quad \underline{y} = \hat{\underline{y}} + \underline{z},$$

der $\hat{\underline{y}} \in W$ og $\underline{z} \in W^\perp$.

Særlig, antar at $\{\underline{v}_1, \dots, \underline{v}_p\}$ er ortogonal basis for W . Da kan vi ta

$$(2) \quad \hat{\underline{y}} = \frac{\underline{y} \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1 + \frac{\underline{y} \cdot \underline{v}_2}{\underline{v}_2 \cdot \underline{v}_2} \underline{v}_2 + \dots + \frac{\underline{y} \cdot \underline{v}_p}{\underline{v}_p \cdot \underline{v}_p} \underline{v}_p.$$

$$\text{og } \underline{z} = \underline{y} - \hat{\underline{y}}.$$

(Skal vise senere at vi alltid kan finne ortogonale baser for W).

Eksempel: $W = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$, underrom av $\mathbb{R}^3$, $\underline{y} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

$\uparrow$ $\uparrow$
 $\underline{v}_1$ $\underline{v}_2$

Finn ortogonal dekomposisjonen fra (1) for $\underline{y}$ og W .

Sjekk: $\underline{v}_1 \cdot \underline{v}_2 = 0$ ✓

$$\text{Fra (2): } \hat{\underline{y}} = \frac{\underline{y} \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1 + \frac{\underline{y} \cdot \underline{v}_2}{\underline{v}_2 \cdot \underline{v}_2} \underline{v}_2$$

$$\underline{y} \cdot \underline{v}_1 = 1 \cdot 2 + 3 \cdot (-1) + 2 \cdot 1 = 1, \quad \underline{y} \cdot \underline{v}_2 = 1 \cdot 1 + 3 \cdot 1 + 2 \cdot (-1) = 2$$

$$\underline{v}_1 \cdot \underline{v}_1 = 2^2 + (-1)^2 + 1^2 = 6, \quad \underline{v}_2 \cdot \underline{v}_2 = 1^2 + 1^2 + (-1)^2 = 3.$$

$$\Rightarrow \hat{\underline{y}} = \frac{1}{6} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/6 \\ 1/6 \end{bmatrix} + \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix} = \text{proj}_W \underline{y}.$$

$$\text{Sett } \underline{z} = \underline{y} - \hat{\underline{y}} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5/2 \\ 5/2 \end{bmatrix}.$$

(Kan sjekke $\underline{z} \in W^\perp$.
 Noe at $\underline{z} \cdot \underline{v}_1 = 0$ og $\underline{z} \cdot \underline{v}_2 = 0$.
 Men $\underline{z} \cdot \underline{v}_1 = \frac{5}{2} - \frac{5}{2} = \underline{z} \cdot \underline{v}_2 = 0$.)

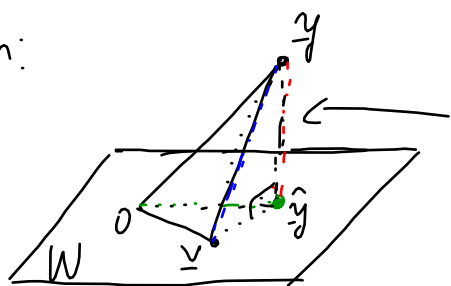
05/10/2026
(3)

Teorem (Best approximation)

Gitt W ^{ikke-triviell} underrom av $\mathbb{R}^n$ og $y \in \mathbb{R}^n$, la $\hat{y} = \text{proj}_W y$ være projeksjonen av y på W . Det gjelder:

$$(3) \quad \|y - \hat{y}\| < \|y - v\|, \text{ for alle } v \in W, v \neq \hat{y}.$$

Illustrasjon:



$$\hat{y} = \text{proj}_W y$$

$$\left. \begin{array}{l} \text{La } z = y - \hat{y}, \|z\| = \|y - \hat{y}\| \\ v, \hat{y} \text{ er i } W \text{ s\aa } \hat{y} - v \in W. \end{array} \right\} \Rightarrow \text{Har } z \cdot (\hat{y} - v) = 0 \text{ (siden } z \in W^\perp)$$

Pythagoras' teorem: $\|z + (\hat{y} - v)\|^2 = \|z\|^2 + \|\hat{y} - v\|^2$ $\|y - v\|$

$$\begin{aligned} z + (\hat{y} - v) &= y - \hat{y} + (\hat{y} - v) = y - v \\ \hat{y} - v &\neq 0, \text{ s\aa } \|\hat{y} - v\| > 0. \end{aligned}$$

Blir: $\|y - v\|^2 = \|z\|^2 + \underbrace{\|\hat{y} - v\|^2}_{>0} > \|z\|^2$

alts\aa: $\|z\| < \|y - v\|$, som ønsket i (3).

Eksempel

$$v_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, y = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, W = \text{span}\{v_1, v_2\} \text{ underrom av } \mathbb{R}^3.$$

Da er $\hat{y} = \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix}$ elementet i W som har kortest avstand til y .

($\hat{y}$ er "nærmest" til y blant alle $v \in W$.)

Eksempel:

$$v_1 = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, y = \begin{bmatrix} -5 \\ -1 \\ 10 \end{bmatrix}. \text{ Finn avstanden fra } y \text{ til } W = \text{span}\{v_1, v_2\}.$$

Løsning: Av teoremet om best approximation, avstanden fra y til W er $\|y - \hat{y}\|$. Regner ut:

$$\hat{y} = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2$$

$$y \cdot v_1 = (-5) \cdot (-2) + (-1) \cdot 5 + 10 \cdot 1 = 15$$

$$v_1 \cdot v_1 = (-2)^2 + 5^2 + 1^2 = 30$$

$$y \cdot v_2 = (-5) \cdot 2 + (-1) \cdot 1 + 10 \cdot (-1) = -21$$

$$v_2 \cdot v_2 = 2^2 + 1^2 + (-1)^2 = 6$$

$$\Rightarrow \hat{y} = \frac{15}{30} v_1 - \frac{21}{6} v_2 = \frac{1}{2} \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} - \frac{7}{2} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \hat{y} = \begin{bmatrix} -8 \\ -1 \\ 4 \end{bmatrix}. \text{ Da er } y - \hat{y} = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix} \text{ og } \|y - \hat{y}\| = \sqrt{3^2 + 0 + (-6)^2} = \sqrt{45}.$$

(Aksutt 6.2) Ortonormale mengder og basiser

Def. En mengde $\{\underline{u}_1, \dots, \underline{u}_p\}$ i $\mathbb{R}^n$ er ortonormal mengde dersom:

$$(1) \quad \underline{u}_j \cdot \underline{u}_k = 0 \text{ for alle } j \neq k, j, k \in \{1, \dots, p\}.$$

$$(2) \quad \underline{u}_j \cdot \underline{u}_j = 1 \text{ for alle } j=1, \dots, p.$$

Merk: (1) sier at mengden er ortogonal. og (2) at elementene $\underline{u}_1, \dots, \underline{u}_p$ er enhetsvektorer (lengde 1).

Merk: Dersom $\{\underline{u}_1, \dots, \underline{u}_p\}$ ortonormal er $\underline{u}_j \neq \underline{0}$ for alle j (fra betingelse (2)).

Spesielt er $\{\underline{u}_1, \dots, \underline{u}_p\}$ en basis for $W = \text{span}\{\underline{u}_1, \dots, \underline{u}_p\}$.

Generelt: En ortonormal basis for underrom W er en basis som er ortonormal mengde.

Eksempel: $\underline{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Normaliserer: $\underline{u}_1 = \frac{1}{\|\underline{v}_1\|} \cdot \underline{v}_1 = \frac{1}{\sqrt{\underline{v}_1 \cdot \underline{v}_1}} \underline{v}_1 = \frac{1}{\sqrt{6}} \underline{v}_1 = \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

$$\underline{u}_2 = \frac{1}{\|\underline{v}_2\|} \cdot \underline{v}_2 = \frac{1}{\sqrt{3}} \underline{v}_2 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}.$$

$\{\underline{u}_1, \underline{u}_2\}$ ortonormal basis for $\text{Span}\{\underline{u}_1, \underline{u}_2\} = \text{Span}\{\underline{v}_1, \underline{v}_2\}$.

Def. En $m \times n$ matrise U kalles ortogonal dersom kolonnene danner en ortonormal mengde.

Teorem 6: U ortogonal matrise hvis og bare hvis $U^T U = I$, identitet av orden n .

"Bevis" for $n=3$, $U = [\underline{u}_1 \quad \underline{u}_2 \quad \underline{u}_3]$ $\underline{u}_1, \underline{u}_2, \underline{u}_3 \in \mathbb{R}^m$

$$U^T = \begin{bmatrix} \underline{u}_1^T \\ \underline{u}_2^T \\ \underline{u}_3^T \end{bmatrix} \quad 3 \times m \text{ matrise}$$

$$U^T U = \begin{bmatrix} \underline{u}_1^T \\ \underline{u}_2^T \\ \underline{u}_3^T \end{bmatrix}_{3 \times m} \cdot \begin{bmatrix} \underline{u}_1 & \underline{u}_2 & \underline{u}_3 \end{bmatrix}_{m \times 3} = \begin{bmatrix} \underline{u}_1^T \cdot \underline{u}_1 & \underline{u}_1^T \cdot \underline{u}_2 & \underline{u}_1^T \cdot \underline{u}_3 \\ \underline{u}_2^T \cdot \underline{u}_1 & \underline{u}_2^T \cdot \underline{u}_2 & \underline{u}_2^T \cdot \underline{u}_3 \\ \underline{u}_3^T \cdot \underline{u}_1 & \underline{u}_3^T \cdot \underline{u}_2 & \underline{u}_3^T \cdot \underline{u}_3 \end{bmatrix}_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\{\underline{u}_1, \underline{u}_2, \underline{u}_3\} \text{ ortonormal} \iff \underline{u}_j \cdot \underline{u}_k = 0 \quad j \neq k$$

$$\underline{u}_j \cdot \underline{u}_j = 1, \quad j=1, 2, 3$$

Ex: $U = \begin{bmatrix} 2/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{3} \end{bmatrix}$ 3×2 matrise.