Eksamen i MAT 1120 - Linear algebra.
13/12 - 04.

Løsninger.

1. $(-1,3),\ (0,2),\ (2,1),\ (3,0)$

$$y = \beta_0 + \beta_1 x.$$

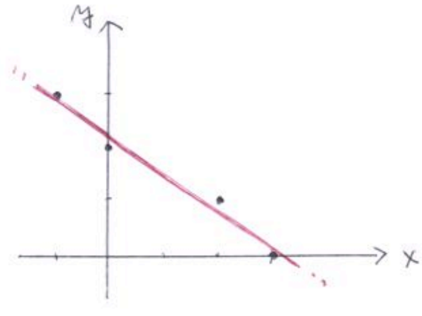
$$X\beta = y, \qquad X = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 2 \\ 1 & 3 \end{bmatrix},\ \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix},\ y = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

$$X^T X\beta = X^T y.$$

$$X^T X = \begin{bmatrix} 4 & 4 \\ 4 & 14 \end{bmatrix}, \quad X^T y = \begin{bmatrix} 6 \\ -1 \end{bmatrix}.$$

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 14 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ -1 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 14 & -4 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ -1 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 88 \\ -28 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 2.2 \\ -0.7 \end{bmatrix}}}$$

$$\underline{\underline{y = 2.2 - 0.7x}}$$



2.
$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} = [a_1\ a_2\ a_3]$$

a)
$$v_1 = a_1 = \underline{\underline{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}}$$

$$v_2 = a_2 - \frac{a_2 \cdot v_1}{v_1 \cdot v_1} v_1 = a_2 - v_1 = \underline{\underline{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}}$$

$$v_3 = a_3 - \frac{a_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{a_3 \cdot v_2}{v_2 \cdot v_2} v_2$$
$$= a_3 - v_1 + \frac{1}{3} v_2 = \underline{\underline{\begin{bmatrix} 2/3 \\ 4/3 \\ -2/3 \end{bmatrix}}}$$

$$u_1 = \frac{v_1}{\|v_1\|} = \underline{\underline{\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}}}, \quad u_2 = \frac{v_2}{\|v_2\|} = \underline{\underline{\begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}}}, \quad u_3 = \frac{v_3}{\|v_3\|} = \underline{\underline{\begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}}}.$$

b)
$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix}, \quad R = Q^T A = \underline{\underline{\begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} & -\frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{4}{\sqrt{6}} \end{bmatrix}}}$$

$$= [u_1\ u_2\ u_3].$$

R er øvre triangulær
<u>med positive diagonalledd.</u>

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3.

a)
$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 8 & 0 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & -\lambda & 2 \\ 0 & 8 & -\lambda \end{vmatrix} = (3-\lambda)(-\lambda)^2 - (3-\lambda)16$$

$$= (3-\lambda)(\lambda^2-16) = (3-\lambda)(\lambda-4)(\lambda+4)$$

Egenverdier $\underline{\underline{\lambda = 3, 4, -4}}$. A er diagonaliserbar siden A har tre forskjellige egenverdier.

Egenrom:

$\underline{\lambda_1 = 3}$.
$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -3 & 2 \\ 0 & 8 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{aligned} x_3 &= 0 \\ -3x_2 + 2x_3 &= 0 \\ 8x_2 - 3x_3 &= 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} &x_2 = x_3 = 0 \\ &x_1 \text{ fri} \end{aligned} \qquad \underline{v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

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$\underline{\lambda_2 = 4}$.

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -4 & 2 \\ 0 & 8 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$$\Leftrightarrow \begin{array}{r} -x_1 \quad + x_3 = 0 \\ -4x_2 + 2x_3 = 0 \\ 8x_2 - 4x_3 = 0 \end{array} \Leftrightarrow \begin{array}{l} x_1 = x_3 = 2x_2 \\ x_2 \text{ fri} \end{array} \qquad \underline{v_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}$$

$\underline{\lambda_3 = -4}$.

$$\begin{bmatrix} 7 & 0 & 1 \\ 0 & 4 & 2 \\ 0 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$$\Leftrightarrow \begin{array}{r} 7x_1 \quad + x_3 = 0 \\ 4x_2 + 2x_3 = 0 \\ 8x_2 + 4x_3 = 0 \end{array} \Leftrightarrow \begin{array}{l} x_1 = -\frac{1}{7}x_3 \\ x_2 = -\frac{1}{2}x_3 \\ x_3 \text{ fri} \end{array} \qquad \underline{v_3 = \begin{bmatrix} -2 \\ -7 \\ 14 \end{bmatrix}}$$

$$P = [v_1\ v_2\ v_3] = \underline{\underline{\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -7 \\ 0 & 2 & 14 \end{bmatrix}}}$$

$$P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

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b)

$$A(c) = \begin{bmatrix} 3 & 0 & 1 \\ 0 & c & 2 \\ 0 & 8 & c \end{bmatrix}, \qquad c \in \mathbb{R}.$$

$$|A(c) - \lambda I| = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & c-\lambda & 2 \\ 0 & 8 & c-\lambda \end{vmatrix} = (3-\lambda)\left[(c-\lambda)^2 - 16\right]$$

$$= (3-\lambda)(\lambda - (c+4))(\lambda - (c-4)).$$

Egenverdier $\lambda = 3,\ c+4,\ c-4$.

Tre forskjellige egenverdier for $c \neq -1, 7$.

Derfor er $A(c)$ diagonaliserbar for $c \neq -1, 7$.

$c=-1$. $\lambda_1 = \lambda_2 = 3$, $\lambda_3 = -5$.

Egenvektorer for $\lambda = 3$:

$$(A(-1) - 3I)\mathbf{x} = \mathbf{0}$$

$$\Leftrightarrow \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & -4 & 2 \\ 0 & 8 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

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$$\Leftrightarrow \quad \begin{aligned} x_3 &= 0 \\ -4x_2 + 2x_3 &= 0 \\ 8x_2 - 4x_3 &= 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} &x_2 = x_3 = 0 \\ &x_1 \text{ fri.} \end{aligned}$$

Egenrommet er av dimensjon 1, mens egenverdien $\lambda = 3$ er av multiplisitet 2. Derfor er matrisen $A(c)$ <u>ikke diagonaliserbar for $c = -1$</u>

<u>$c = 7$</u> $\lambda_1 = \lambda_3 = 3$, $\lambda_2 = 11$.

Egenvektorer for $\lambda = 3$:

$$(A(7) - 3I)\mathbf{x} = \mathbf{0}$$

$$\Leftrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 4 & 2 \\ 0 & 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

$$\Leftrightarrow \quad \begin{aligned} x_3 &= 0 \\ 4x_2 + 2x_3 &= 0 \\ 8x_2 + 4x_3 &= 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} &x_2 = x_3 = 0 \\ &x_1 \text{ fri} \end{aligned}$$

Egenrommet er av dimensjon 1, mens egenverdien $\lambda = 3$ er av multiplisitet 2. Derfor er matrisen $A(c)$ <u>ikke diagonaliserbar for $c = 7$</u>.