Goodstein-sequences and the fight against Hydra

A Goodstein sequence is a sequence of numbers \( \{a_n\}_{n \geq 2} \) where we compute \( a_{n+1} \) from \( a_n \) in the following way:

- Write \( a_n \) in superpolynomial form in base \( n \).
- Replace base \( n \) with base \( n+1 \), but keep the form.
- Subtract 1.

**Example 1** If \( a_2 = 12 \), we write

\[
12 = 2^{2+1} + 2^2
\]

Shift of base gives

\[
a_3 = 3^{3+1} + 3^3 - 1 = 3^{3+1} + 2 \cdot 3^2 + 2 \cdot 3 + 2 = 107.
\]

\[
a_4 = 4^{4+1} + 2 \cdot 4^2 + 2 \cdot 4 + 1 = 1065.
\]

It is a fact that all Goodstein sequences will terminate in the sense that they will eventually take the value 0 (prove this) but it is also a fact that this cannot be proved without introducing the theory of infinite sets (do not prove this).

HYDRA is originally a creature from Greek mythology, but we will consider a mathematical model.

To us, a Hydra is a finite tree that may be pruned by cutting off leaf nodes, but that to each pruning responds by growing a finite set of new branches, \( n \) new subtrees the \( n \)’th time we cut off one leaf node. If we cut off one leaf node, the growing point will be the node two levels below, and the new subtrees will all be copies of the remaining tree above that node.

It is a fact that any Hydra will eventually be pruned down to the root independent on how we choose the leaf nodes, but it is also a fact that this cannot be proved in elementary number theory (first order Peano arithmetic).

In this project, you will be asked to prove the two easy facts.

The main task will be to show that the two statements are equivalent modulo elementary number theory.

Intimate knowledge of logic is not necessary for working with this project.

Dag Normann
Transfinite sequences of Gödel sentences.

Gödel’s incompleteness theorem says that if a formal theory $T$ is axiomatizable and sufficiently strong for elementary number theory to be reducible to $T$, and for the natural numbers to be a model for $T$, then there is a universal sentence $A$ in number theory that is true in $\mathbb{N}$, but that is not provable from the axioms of $T$.

If we ad $A$ as an axiom, we get a new axiomatizable theory $T, A$ and the theorem can be used again. This can of course be repeated infinitely many times, but does it make sense, and is it possible, to go beyond that?

Given an axiomatizable theory $T$ satisfying the assumption of the incompleteness theorem, we have a natural preordering on the set of universal sentences $A$ and $B$, by $A \leq_T B$ if $A$ is provable in the theory $T, B$. In this project we will investigate this ordering. One property is that there will be transfinite, strictly increasing sequences of universal sentences, where, in a sense, each sentence is the Gödel sentence of $T$ extended with all predecessors.

In order to work on this project, you should know something about computable functions, for instance defined on the basis of Turing machines. It is not essential to know the details of the proof of the incompleteness theorem, but some familiarity with the argument might be of help.

Dag Normann
Games and computable winning strategies

We will consider games where players I and II in alternating moves pick finite binary sequences, and where they as the result of concatenating these sequences construct an infinite sequence of zeros and ones. Let $X$ be the set of infinite binary sequences, and let $A \subseteq X$. I wins the game $G_A$ if the result of the game is in $A$, otherwise II wins. Of course, these games will last infinitely long, but as mathematical entities they are sound.

A game like this is called open if for all $\{x_i\}_{i \in \mathbb{N}} \in A$ there is an $n \in \mathbb{N}$ such that all continuations of $x_1, \ldots, x_n$ to an infinite sequence will be in $A$. We then say that player I has won in position $x_1, \ldots, x_n$.

If we identify the set of binary sequences with the Cantor set on $[0, 1]$, a game $G_A$ is open if and only if $A$ is the intersection of an open set and the Cantor space.

A strategy for a player will be a function that to any position in a game, where the player is in the move, gives a possible next move. A winning strategy for a player is a strategy such that the player will win independent of how the opponent plays as long as the player follows the strategy.

First show that it is impossible that both players have a winning strategy, and that for any open game, one of the players will have a winning strategy. (This theorem actually holds when the set $A$ is so called Borel, but do not try to prove this.) Then find examples on how some popular real-life games can be digitalized to games of the form described above. Note that there is no “draw” or “remis” in these games, for player II, “winning” is what we normally would call “not losing”.

Kleene constructed a decidable binary tree that is infinite, but where there are no infinite, computable branches. Use this to construct an open game where player II will have a winning strategy, but no computable winning strategy. The set should be decidable in the sense that the set of winning positions for both players should be decidable.

One needs to know a little bit about Turing machines and the unsolvability of the halting problem for this project.

Dag Normann