1 Path-wise integration

In the last 20 years, the theory of rough paths has received much attention from researchers in fields of stochastic analysis, algebra, and statistics. The theory presents an alternative approach to stochastic integration, as well as a characterisation of statistical feature’s of time series data in a non-parametric way. Especially important applications has been towards the theory of machine learning, mathematical physics, non-parametric statistics, as well as finance and risk (see e.g. [1, 3, 2, 7]). Furthermore, this theory constitutes the foundation that 2014 Fields medalist Martin Hairer used in order to create his abstract "theory of regularity structures" [6], which provides an algebraic and analytic framework in order to analyse singular partial differential equations. Such equations is frequently appearing in quantum field theory, but also have several other applications.

The first fundamental question in this direction relates to the construction of integrals along irregular paths. By this we mean the following; consider two functions $X: [0, T] \to \mathbb{R}^d$ and $Y: [0, T] \to \mathbb{R}^d \times \mathbb{R}^d$. We call $X$ the driving signal or noise, and $Y$ the integrand. We are interested in the integral

$$Z_t - Z_s := \int_s^t Y_s dX_s.$$  \hspace{1cm} (1)

Of course, in order to make sense of this integral, we need some assumptions on $X$ and $Y$. For example, if $X$ is a differentiable function, and $Y$ is continuous, then we have

$$Z_t - Z_s = \int_s^t Y_s \dot{X}_s ds,$$  \hspace{1cm} (2)

where $\dot{X}$ is the derivative of $X$, and the integral can be interpreted as a standard Riemann integral. However, our objective is to study these integrals when the driving signal is no-where differentiable. Typically, such paths possess some sort of fractal like behavior, or self similarity. In Figure 1 and 2 in Section 4 we see two examples of such paths, namely the sample path of a Brownian motion and the fractal behavior of Weierstrass function. In the case when $X$ is a Brownian motion, one could use the probabilistic properties of $X$ in order to give a definition of the integral in this case. On the other hand, we are interested in constructing the integral only using the analytic properties of $X$, i.e. the regularity of $X$.

The first objective of the project will be to show how one can construct this integral in the case when $X$ and $Y$ are Hölder continuous functions. That is, these functions satisfy for all $s, t \in [0, T]$

$$|X_t - X_s| \leq C|t - s|^\alpha \quad \text{and} \quad |Y_t - Y_s| \leq C|t - s|^\beta,$$  \hspace{1cm} (3)

where $\alpha$ and $\beta$ is in the open interval $(0, 1)$ and $\alpha + \beta > 1$. The integral we will obtain is known as the Young integral, and will be constructed purely from deterministic analytical tools (no stochastic analysis involved at this point). A main tool for this construction will be the celebrated Sewing lemma,
which can be found in [4, Lemma 4.2].

The second objective is to investigate the analytic properties of this integral when under different examples of driving noise $X$ and integrand $Y$. In particular, assuming $X$ is a sample path of a Gaussian stochastic process, of sufficient Hölder regularity, and $Y$ a deterministic path, we will show that for each $t \in [0,T]$, the integral $Z_t = \int_0^t Y_r dX_r$ is in fact a Gaussian random variable.

There are several introductory books on integration theory for irregular functions. We propose to at least consider the references [4], [5] and [8]. More precise section numbers to study and important lemmas will be provided later.

2 Controlled differential equations

This part will only be considered if time permits.

We will investigate differential equations controlled by Hölder continuous noise. These equations can formally be written as

$$\dot{Y}_t = f(Y_t)\dot{X}_t, \quad Y_0 \in \mathbb{R}^d.$$  \hspace{1cm} (4)

Note that since $X$ is only Hölder continuous, the object $\dot{X}_t$ does not really exists. However, integrating with respect to time on each side of the equation above, we can consider the equation on its integral form

$$Y_t = Y_0 + \int_0^t f(Y_r)dX_r.$$  \hspace{1cm} (5)

We see that if $X$ and $Y$ are regular enough, and $f$ is a smooth function, it is reasonable to believe that the integral appearing on the right hand side of the equation, may be interpreted as a Young integral, such as constructed in the first part of the project.

We will show existence and uniqueness of equations on this form, when $X$ and $Y$ are Hölder continuous paths, and show the stability of the solution with respect to the driving noise $X$. The main literature for this section can be found in [4, Section 8].

3 Summary

In summary, the project will go through the following steps:

1 We review the concept of Hölder continuous functions, and also the Riemann integral.

2 We will prove that there the Young integral given in (1) exists when the functions $X$ and $Y$ are Hölder continuous, satisfying a simple criterion.

3 We look at examples of Hölder continuous functions such as the Brownian motion or the Weierstrass function, and construct the integral in this case.

If time permits, we will also consider one two of the following topics:

4 Given that the signal $X$ in (1) is a Brownian motion and $Y$ is a square integrable deterministic function, we will show that the integral in (1) is a Gaussian random variable.

5 We will study differential equations controlled by Hölder continuous noise, as discussed in Section 2.

The project is mainly concerned with real-analysis, with possibly some aspects of probability theory, and linear algebra.
References


4 Appendix

Figure 1: Sample path of Brownian motion
Figure 2: The Weierstrass function