

Expander graphs from property (T) groups

How do we get resilient sparse graphs?

Makoto Yamashita

Aim: The aim of this project is to learn about interesting connection between analysis and combinatorics, through a famous application of the theory of spectral gaps for graphs and discrete groups to construction of large graphs which are hard to break up.

Background: Let G be a graph with vertices V and edges E . For our purpose, this means that V and E are (finite) sets, together with a decomposition $E = \bigsqcup_{\{v,w\}} E_{\{v,w\}}$ for v, w in V . We interpret $E_{\{v,w\}}$ as the set of edges between v and w , and we do not distinguish the labeling $\{v,w\}$ and $\{w,v\}$ (edges do not have orientation).

If X is a subset of V , we define the boundary ∂X of X as the union of $E_{\{v,w\}}$ such that v is in X and w is in the complement $V \setminus X$. Based on this, the quantity

$$h(G) = \min_{\emptyset \neq X \subseteq V} \frac{|\partial X|}{\min\{|X|, |V \setminus X|\}},$$

called the *Cheeger constant* of G , measures how hard it is to separate G into two disjoint parts by cutting edges. We want to construct *expander graphs*, a sequence of graphs $G^{(1)}, G^{(2)}, \dots$ with $V^{(n)}$ growing to ∞ while bounding the number of edges connected to vertices, but $h(G^{(n)})$ is bounded away from 0. This corresponds to expanding graphs with certain resilience for cutting.

The structure of graph G can be encoded by the adjacency matrix A defined as $A_{\{v,w\}} = |E_{\{v,w\}}|$, which is a square integer-valued matrix with index set V . Assume moreover that each v is connected to k edges (G is k -regular), and consider $\Delta = k I_V - A$, the *graph Laplacian*. This matrix is positive semidefinite, and the dimension of its kernel is equal to the number of connected components of G (which can be assume to be 1).

There are series of inequalities (*isoperimetric inequalities*) which relate $h(G)$ to the smallest nonzero eigenvalue $\lambda_1(G)$ of Δ . Since 0 is the smallest eigenvalue of Δ , $\lambda_1(G) = \lambda_1(\Delta) - 0$ is also called the *spectral gap* of G . As it happens, the spectral gap of graph Laplacian is tightly related to the notion of *property (T)* for discrete (or locally compact) groups introduced by Kazhdan in his study of lattices in SL_n and other higher rank Lie groups. This connection was discovered by Margulis in 1975, who showed that Cayley graphs of finite quotients of a group with property (T) form expander graphs. This led to very fruitful interaction between the graph theory and noncommutative harmonic analysis (unitary representation of noncommutative groups).

What we do: We begin by going through materials in Lubotzky's book, in particular Chapters 1 and 4. This should be enough to learn the basic general idea of this subject. As for the concrete example of expander graphs, we plan to draw them from the property (T) of $SL_3(\mathbb{Z})$ following the original paper by Kazhdan, or more elementary proofs following Shalom and Ozawa.

References

Main reference: Alexander Lubotzky, *Discrete groups, expanding graphs and invariant measures*, Progress in Mathematics, vol. 125, Birkhäuser Verlag, Basel, 1994, With an appendix by Jonathan D. Rogawski.

Further reading:

- Nathaniel P. Brown and Narutaka Ozawa, *C*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, Amer. Math. Soc., Providence, RI, 2008.
- D. A. Kazhdan, *On the connection of the dual space of a group with the structure of its closed subgroups*, Funkcional. Anal. i Priložen. **1** (1967), 71–74.
- Yehuda Shalom, *The algebraization of Kazhdan's property (T)*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 1283–1310.