

A FAMILY OF HILBERT-TYPE INEQUALITIES

OLE FREDRIK BREVIG

This project is related to the following question: Fix a positive real number α . What is the smallest real number $C_\alpha > 0$ such that the inequality

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m a_n \frac{(mn)^{\alpha-1/2}}{(\max(m, n))^{2\alpha}} \leq C_\alpha \sum_{m=1}^{\infty} |a_m|^2$$

holds for every sequence of real numbers $a = (a_1, a_2, \dots)$? In the range $0 < \alpha \leq 1$ it has been known for over a century that $C_\alpha = 2/\alpha$. I have investigated this family of inequalities in a series of recent papers due to their relevance for problems in operator theory and spectral analysis. For example, I can prove that $C_\alpha = 2/\alpha$ holds for $0 < \alpha \leq 1.5$ and that $C_\alpha = 2/\alpha$ does not hold for $\alpha \geq 1.7$.

The two main techniques used to estimate C_α are classical: The Schur test and Euler–Maclaurin summation¹. Take a look at the material in Chapter 10 of [3] and Chapter 11.5 of [2]. You can also check out my first paper on the topic [1].

This is a project for those that prefer concrete computations over abstract theories. The first part of the project will be devoted to studying the two main techniques. The second part is to investigate inequalities such as the family presented above. The scope is fairly flexible and can be adjusted according to the student's ability and ambition. It may even be possible to go beyond the state-of-the-art and establish new results of sufficient interest to warrant publication in a peer-reviewed journal.

Recommended previous knowledge. MAT1000, MA1120, MAT2400. Basic programming skills (e.g. Python/SciPy) may come in handy.

REFERENCES

1. Ole Fredrik Brevig, *Sharp norm estimates for composition operators and Hilbert-type inequalities*, Bull. Lond. Math. Soc. **49** (2017), no. 6, 965–978.
2. Peter Duren, *Invitation to classical analysis*, Pure and Applied Undergraduate Texts, vol. 17, American Mathematical Society, Providence, RI, 2012.
3. J. Michael Steele, *The Cauchy-Schwarz master class*, AMS/MAA Problem Books Series, Mathematical Association of America, Washington, DC; Cambridge University Press, Cambridge, 2004, An introduction to the art of mathematical inequalities.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSLO, 0851 OSLO, NORWAY
Email address: obrevig@math.uio.no

¹Euler invented this summation method in order to obtain so precise numerical estimates for the slowly convergent series $\sum_{n \geq 1} \frac{1}{n^2}$ that he was able to guess that its sum is $\frac{\pi^2}{6}$. Euler then proceeded to prove his guess rigorously, thereby solving the *Basel problem* and becoming famous.