

Numerical estimate of random matrices

What is the shape of random matrix?

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Aim: The aim of this project is to get quantitative understanding of random matrices. After learning minimum prerequisites on random matrices, we will do numerical simulation to check and visualize main results of the theory.

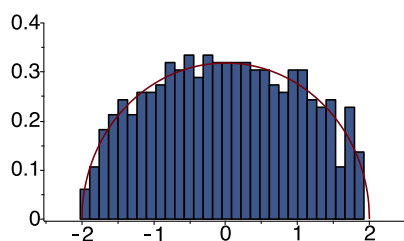
Background: A *random matrix* is a matrix-valued random variable. One of the pioneering results in this theory is due to Wigner (1958), which is about Hermitian random matrix with independent upper triangular components. Given such a random matrix X^N of size N , its eigenvalues $\lambda_i(X^N)$ give random variables in the usual sense. Wigner's theorem states that, when one lets N grow to infinity, the averaging with respect to these random numbers converges to the integration with *semicircular distribution* under a mild condition: if $f(x)$ is a function in one real variable,

$$\frac{1}{N} \sum_{i=1}^N f(\lambda_i(X^N)) \rightarrow \int_{-2}^2 f(x) \frac{\sqrt{4-x^2}}{2\pi} dx \quad (N \rightarrow \infty) \quad \text{almost surely.}$$

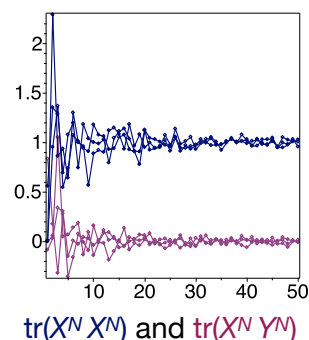
Another major development of the theory happened in the late 80's, through the invention of *free probability* theory by Voiculescu. He considered several sequences of independent random matrices, and discovered a new notion of independence, called the *free independence*, in the framework of noncommutative probability theory. This is a field where noncommutative algebra meets probability theory, and it provided many important application to the structure of algebras of operators on infinite dimensional inner product spaces (*Hilbert spaces*) besides understanding multivariate random matrices. Moreover, Speicher, Nica, and others extracted a interesting combinatorial structure, called *noncrossing partitions*, behind the computation of multivariate moments in free probability.

What we do: We will begin by learning the basics of random matrices. Here, we just try to understand the statements of important results, without too much going into the proofs.

We then start numerical simulation and visualization with Maple, starting from Wigner's theorem. We next move to Voiculescu's theorem, and compute cumulants for several series of random matrices. We will observe how the *genus expansion* works to isolate the noncrossing partitions. Throughout the project, the important thing is to understand the asymptotic behavior of various quantities, such as the expected higher moments, as the size of matrices goes to infinity.



eigenvalue distribution of
GUE random matrix



References

Main reference: Alice Guionnet, *Large random matrices: lectures on macroscopic asymptotics*, Lecture Notes in Mathematics, vol. 1957, Springer-Verlag, 2009.

Further reading: Alexandru Nica and Roland Speicher, *Lectures on the combinatorics of free probability*, London Mathematical Society Lecture Note Series, vol. 335, Cambridge University Press, 2006.