Ramanujan matrixes from interlacing polynomials

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Goal

- Low-level: understand the application of interlacing polynomials to spectral graph theory
- High-level: witness an interesting relation between combinatorics, probability theory, and linear algebra through this reading

Description

This is a reading project on the application of interlacing polynomials to spectral graph theory, in particular, the Ramanujan graphs. *Expander graphs* form an important class of examples in the spectral graph theory, which studies various properties of graphs through the eigenvalues of their adjacency matrices. Let Γ be a d-regular graph, that is, each vertex is connected to d edges. If we take its adjacency matrix A_Γ and its eigenvalues $\lambda_1(A_\Gamma)$, $\lambda_2(A_\Gamma)$, $\ldots$ arranged in the decreasing order, we have the largest eigenvalue $\lambda_1(A_\Gamma) = d$ (from knowing that $(1, \ldots, 1)^T$ is an eigenvector of A_Γ), while the second eigenvalue $\lambda_2(A_\Gamma)$ is related to the *Cheeger constant* $h(\Gamma)$ of Γ, which gives an estimate on the size of subsets of vertices and their boundary sets.

One fundamental result of the theory (the Alon–Boppana bound) says that, if $(\Gamma^{(i)})_{i=1}^{\infty}$ is a sequence of d-regular graphs, then we have $\liminf_{i\to\infty} \lambda_2(A_{\Gamma^{(i)}}) \geq 2\sqrt{d-1}$. We say that $(\Gamma^{(i)})_i$ is a sequence of *Ramanujan graphs* if we have $\lambda_2(A_{\Gamma^{(i)}}) \leq 2\sqrt{d-1}$, so that the second largest eigenvalue $\lambda_2(A_{\Gamma^{(i)}})$ stays at the optimal level. Such graphs is a class of expander graphs, and represent mathematical models of network systems that are resilient to cutting.

It turned out that constructing examples of Ramanujan graphs is a highly nontrivial task, and the first constructions due to Margulis and Lubotzky–Phillips–Sarnak relied on deep connections between eigenvalues of graphs coming from groups and geometry behind them, which forced d to be of the form $p^k + 1$ for a prime number p. In a remarkable paper [MSS15], Marcus, Spielman, and Srivastava found a new way to construct Ramanujan graphs for arbitrary d based on the theory of interlacing polynomials, which look at real polynomials $f(t)$ and $g(t)$ such that their roots $\lambda_1, \ldots, \lambda_k$ and $\mu_1, \ldots, \mu_{k-1}$ form an interlacing sequence, that is, $\lambda_1 > \mu_1 > \lambda_2 > \ldots$.

Our goal is to read a recent survey [Sri24] by Srivastava, one of the authors of [MSS15] to understand the main ideas behind the proof. While this survey offers a good outline of the theory, we will take up extra materials such as [MSS15] in order to pick up necessary details.

References

[MSS15] Adam W. Marcus, Daniel A. Spielman, and Nikhil Srivastava, *Interlacing families I: Bipartite Ramanujan graphs of all degrees*, Ann. of Math. (2) **182** (2015), no. 1, 307–325.

[Sri24] Nikhil Srivastava, *Ramanujan graphs and interlacing families*, preprint, 2024. (`arXiv:2412.20721`)