

The rank-nullity ring of a matroid

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Abstract

We introduce the *rank-nullity ring* of a matroid M . This is the subring of the Chow ring of the free matroid generated by sums of variables corresponding to subsets having the same rank and the same nullity in the matroid M . We present symmetric combinatorial expressions for the tautological Chern classes of M , introduced by Berget, Eur, Spink and Tseng which imply that they live in the rank-nullity ring of M .

Matroids and the Chow ring

A **matroid** on a ground set E of rank d is a function $r : 2^E \rightarrow \mathbb{Z}$ satisfying:

- $0 \leq r(S) \leq |S|$
- $S \subset U$ implies $r(S) \leq r(U)$
- $r(S \cup U) + r(S \cap U) \leq r(S) + r(U)$

A **flat** is a subset $F \subseteq E$ such that $r(F \cup \{i\}) \geq r(F)$ for all $i \in E \setminus F$. The **Chow ring** [FY04] of a matroid is given by:

$$A^*(M) = \frac{\mathbb{Z}[x_F \mid \emptyset \subsetneq F \subset E]}{\langle x_F x'_F \mid F \not\subseteq F' \text{ and } F \not\supseteq F' \rangle + \langle \sum_{i \in F} x_F \mid i \in E \rangle}.$$

The rank-nullity ring of a matroid

The **uniform matroid** $U_{d,n}$ is given by:

$$r(S) = \begin{cases} |S| & \text{if } |S| \leq d \\ d & \text{else.} \end{cases}$$

Definition (Fife M. Rincón)

The **rank-nullity** ring R_M of a matroid M is given by:

$$R_M := \mathbb{Z}[y_{i,j} \mid i, j \in [0, n], i + j \leq n] \subseteq A^*(U_{n,n}),$$

where $y_{i,j} = \sum_{r(S)=i, \text{null}(S)=j} x_S$ and $\text{null}(S) = |S| - r(S)$.

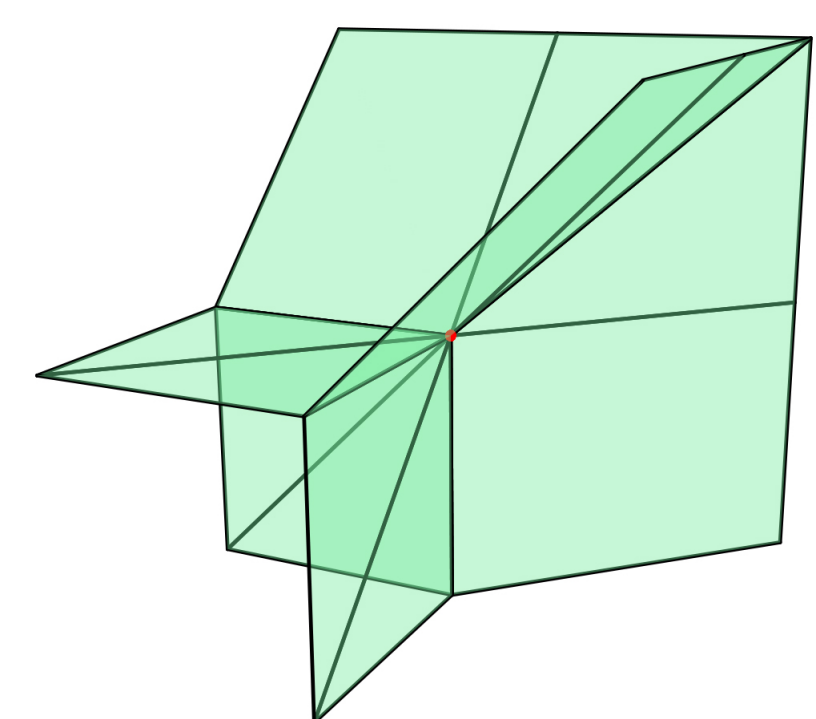
Chern numbers of matroids (motivation)

To a matroid we can associate the **Ardila-Klivans fan** Σ_M . The ring of **Minkowski weights** $MW_*(M)$ consists of weighted subfans satisfying the balancing condition.

Theorem ([AHK18]).

$$MW_*(M) \simeq A^*(M).$$

The **Chern numbers** [Man23] of a matroid are obtained by intersecting the **Chern-Schwartz-MacPherson cycles** [LRS20] of a matroid. These are elements of $MW_*(M)$. To better understand these numbers we want an expression for the CSM cycles of a matroid in $A^*(M)$.



$$\begin{array}{ccc} MW_k(M) & \xrightarrow{\simeq} & A^{d-k}(M) \\ \Downarrow & & \Downarrow \\ csm_k(M) & \longrightarrow & ? \end{array}$$

The Chern classes in the Chow ring

The CSM cycles are tightly related to a new geometric framework: the **tautological class of a matroid** [Ber+23] S_M , which unifies many matroid invariants. Its Chern classes satisfy

$$c_k(S_M) \in A^*(U_{n,n}).$$

Theorem ([Ber+23]).

$$csm_k(M) = c_{d-1-k}(S_M) \cdot \Delta_M,$$

where Δ_M is the class of the Ardila-Klivans fan.

Theorem (Fife M. Rincón)

The Chern classes $c_k(S_M)$ of a matroid are given by

$$c_k(S_M) = \sum_{\emptyset \not\subseteq S_1 \subsetneq \dots \subsetneq S_t \subseteq E} \alpha_S(S_1^{p_1}, \dots, S_t^{p_t}) x_{S_1}^{p_1} x_{S_2}^{p_2} \dots x_{S_t}^{p_t},$$

where $\sum_{i=1}^t p_i = k$ and the sum is taken over all chains of subsets.

The coefficients are given by

$$\alpha_S(S_1^{p_1}, S_2^{p_2}, \dots, S_t^{p_t}) = (-1)^k \binom{r(S_1)}{p_1} \binom{r(S_2) - \tilde{p}_1}{p_2} \dots \binom{r(S_t) - \tilde{p}_{t-1}}{p_t}$$

where $\tilde{p}_i = \sum_{j=1}^i p_j$.

Corollary (Fife M. Rincón). The Chern classes $c_k(S_M)$ live in the rank-nullity ring R_M .

Properties of R_M

Proposition (Fife M. Rincón).

- $R_M \subseteq A^*(U_{n,n})^{\mathbf{Aut}(M)}$
- $R_{U_{d,n}} = A^*(U_{n,n})^{\mathcal{S}_N}$

Corollary (Fife M. Rincón)

The set

$\mathcal{B}_{R_{U_{d,n}}} = \{y_{i_1}^{p_1} y_{i_2}^{p_2} \dots y_{i_l}^{p_l} \mid i_j - i_{j-1} > p_j - p_{j-1} \text{ and } i_1 < i_2 < \dots < i_l\}$ is a basis for $R_{U_{d,n}}$

$$\left[\binom{n-1}{0}, \binom{n-1}{1}, \dots, \binom{n-1}{n-1} \right].$$

Proposition (Fife M. Rincón). The Hilbert function of R_M is non-decreasing up to $\lfloor \frac{n}{2} \rfloor$.

Bibliography

[FY04]

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