

Exact Johnson solids in OSCAR

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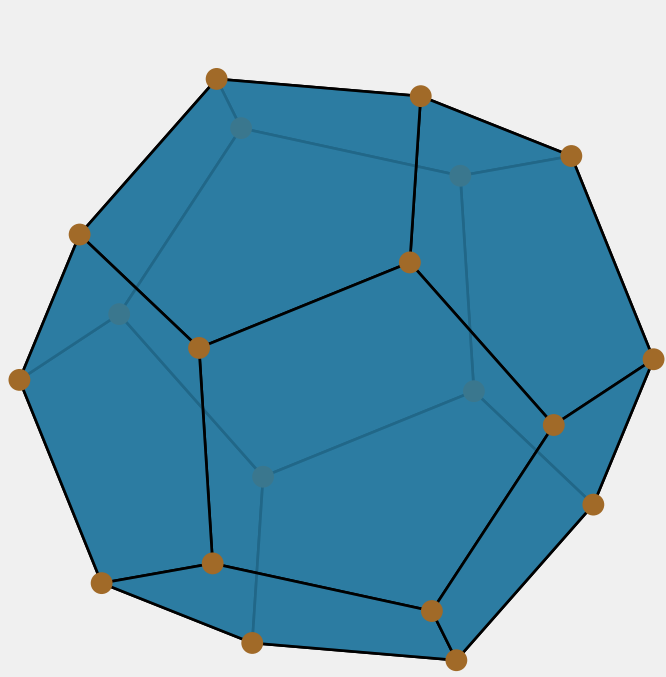


Solids with regular faces

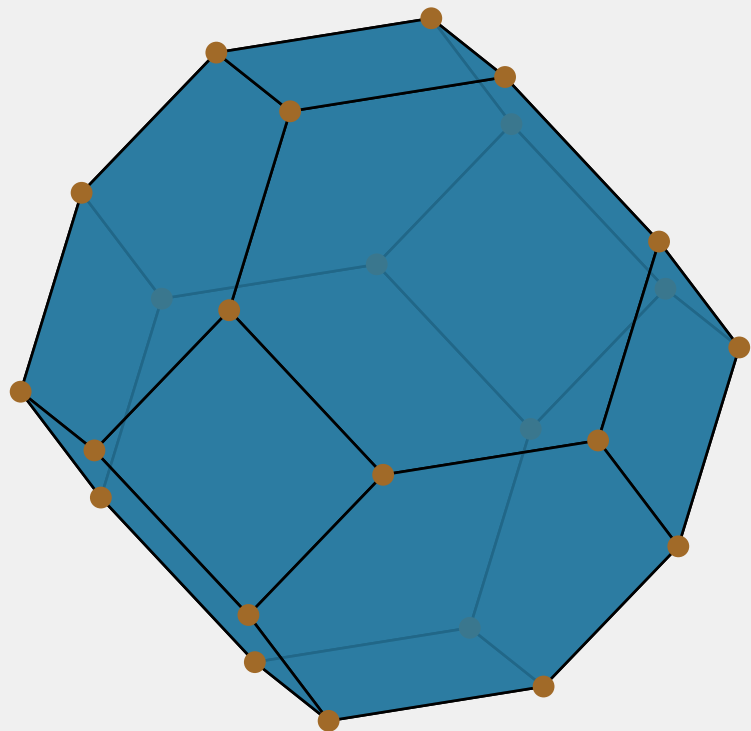
The following polytopes make up all 3-dimensional polytopes with regular faces:

- 5 Platonic solids: vertex- and facet-transitive,
- 13 Archimedean solids: vertex-transitive,
- Regular prisms and antiprisms,
- 92 Johnson solids: not vertex-transitive.

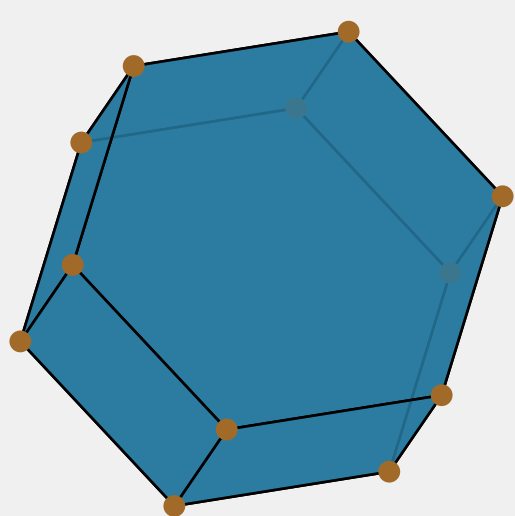
A polytope is *vertex-transitive* if all its vertices are equivalent under its symmetries.



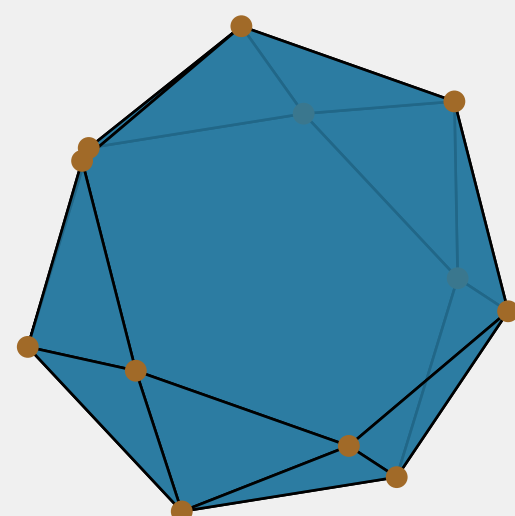
The dodecahedron



The permutohedron



The regular prism over a hexagon



The regular antiprism over a hexagon

Johnson solids

A list of the 92 Johnson solids was computed by Johnson in 1966 [5] and proven to be complete by Zalgaller in 1969 [7].

Johnson solids can be created by e.g.

- Elongations and gyroelongations: adding a prism or antiprism,
- Augmentations: adding a pyramid,
- Diminishings: taking away a pyramid or cupola,

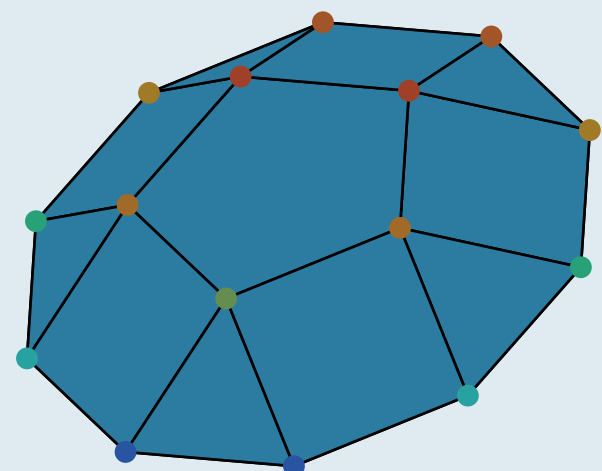
from (bi)pyramids and (bi)cupolas or the other regular solids.

There are 9 elementary or non-composite Johnson solids.

The exact coordinates of most Johnson solids include non-rational algebraic numbers. Some of them use roots of complicated polynomials and are notoriously hard to compute.

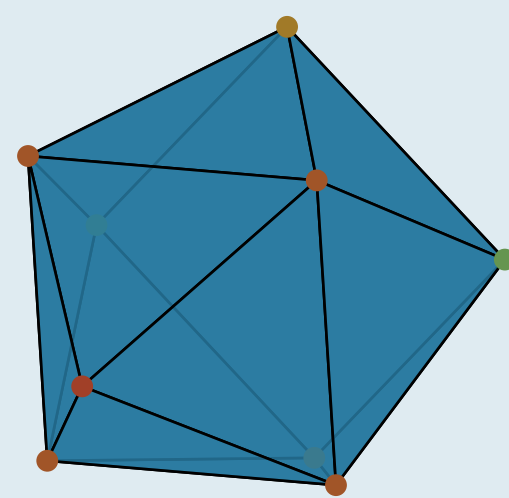
We implemented the 63 non-rational ones in OSCAR [2], verifying or correcting existing data.

For the non-composite Johnson solid we followed Timofeenko's computations [6].



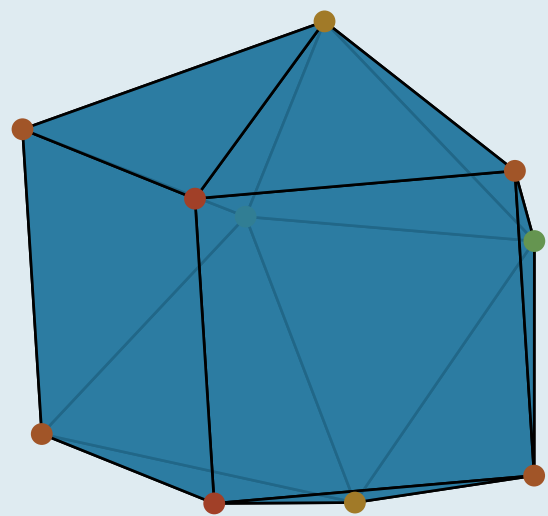
The pentagonal cupola J(5):
Representable by a simple quadratic algebraic field extension over $\mathbb{Q}$.

$$\begin{aligned} & \bullet \left(\pm \frac{1}{2}, \frac{11}{60}\sqrt{5} + \frac{1}{4}, -\frac{13}{60}\sqrt{5} + \frac{1}{12} \right) \\ & \bullet \left(\pm \frac{1}{2}, \frac{11}{60}\sqrt{5} + \frac{1}{4}, -\frac{13}{60}\sqrt{5} - \frac{11}{12} \right) \\ & \bullet \left(\pm \left(\frac{1}{4}\sqrt{5} + \frac{1}{4} \right), \frac{11}{60}\sqrt{5} - \frac{1}{4}, \frac{1}{30}\sqrt{5} + \frac{1}{3} \right) \\ & \bullet \left(\pm \left(\frac{1}{4}\sqrt{5} + \frac{3}{4} \right), -\frac{1}{15}\sqrt{5} + \frac{1}{2}, -\frac{13}{60}\sqrt{5} - \frac{5}{12} \right) \\ & \bullet \left(0, -\frac{1}{15}\sqrt{5}, \frac{1}{30}\sqrt{5} + \frac{5}{6} \right) \\ & \bullet \left(\pm \left(\frac{1}{2}\sqrt{5} + \frac{1}{2} \right), -\frac{1}{15}\sqrt{5}, \frac{1}{30}\sqrt{5} - \frac{1}{6} \right) \\ & \bullet \left(\pm \left(\frac{1}{4}\sqrt{5} + \frac{3}{4} \right), -\frac{1}{15}\sqrt{5} - \frac{1}{2}, \frac{17}{60}\sqrt{5} + \frac{1}{12} \right) \\ & \bullet \left(\pm \frac{1}{2}, -\frac{19}{60}\sqrt{5} - \frac{1}{4}, \frac{17}{60}\sqrt{5} + \frac{7}{12} \right) \end{aligned}$$



The gyroelongated square pyramid J(10):
Representable by a simple non-quadratic algebraic field extension over $\mathbb{Q}$.

$$\begin{aligned} & \bullet \left(0, 0, \frac{1}{4}\sqrt{8} + \frac{1}{4}\sqrt[4]{8} \right), \\ & \bullet \left(\pm \frac{1}{2}, \pm \frac{1}{2}, \frac{1}{4}\sqrt[4]{8} \right), \\ & \bullet \left(0, \pm \frac{1}{4}\sqrt{8}, -\frac{1}{4}\sqrt[4]{8} \right), \\ & \bullet \left(\pm \frac{1}{4}\sqrt[4]{8}, 0, -\frac{1}{4}\sqrt[4]{8} \right) \end{aligned}$$



The spenocorona J(86): Representable
by a non-simple algebraic field extension
over $\mathbb{Q}$

$$\begin{aligned} & \bullet \left(0, \pm \frac{1}{2}, b \right) \\ & \bullet \left(\pm a, \pm \frac{1}{2}, 0 \right) \\ & \bullet \left(0, \pm \left(\frac{2}{27}a^3 + \frac{32}{135}a^2 + \frac{334}{405}a - \frac{107}{810} \right), \left(\frac{16}{15}a^3 + \frac{106}{75}a^2 - \frac{568}{225}a - \frac{83}{225} \right) b \right) \\ & \bullet \left(\pm \frac{1}{2}, 0, \left(\frac{13}{27}a^3 + \frac{208}{135}a^2 - \frac{1733}{810}a - \frac{449}{405} \right) b \right) \end{aligned}$$

Here $a \approx 0.85$ is a root of the polynomial $f \in \mathbb{Q}[x]$

$$f(x) = 60x^4 - 48x^3 - 100x^2 + 56x + 23$$

and $b \approx 0.52$ is a root of $g \in \mathbb{Q}(a)[y]$

$$g(y) = y^2 + a^2 - 1.$$

We used OSCAR's functionality of algebraic field extensions. In OSCAR non-simple and non-quadratic algebraic field extensions are possible.

Use the Johnson solids in OSCAR

```
julia> J86 = johnson_solid(86)
Polytope in ambient dimension 3 with
  EmbeddedNumFieldElem{Hecke.RelSimpleNumFieldElem{AbsSimpleNumFieldElem}}
  type coefficients

julia> vertices(J86)[7]
3-element PointVector{EmbeddedNumFieldElem{Hecke.RelSimpleNumFieldElem{
  AbsSimpleNumFieldElem}}}:
 0 (0.00)
 2//27*a^3 + 32//135*a^2 + 334//405*a - 107//810 (0.79)
 (16//15*a^3 + 106//75*a^2 - 568//225*a - 83//225)*b (-0.43)

julia> volume(J86)
(-13//6*a^3 + 17//30*a^2 + 653//180*a + 133//180)*b (1.52)

julia> MILP = mixed_integer_linear_program(J86, [-1,1,1]; integer_variables = [3]);

julia> solve_milp(MILP)
((a + 1//2),
  EmbeddedNumFieldElem{Hecke.RelSimpleNumFieldElem{AbsSimpleNumFieldElem}}
  [-a, 1//2, 0])

julia> F = coefficient_field(J86)
Embedded field
  Relative number field of degree 2 over number field
  at
  Complex embedding corresponding to root 0.52 of relative number field

julia> defining_polynomial(number_field(F))
y^2 + a^2 - 1

julia> defining_polynomial(base_field(number_field(F)))
60*x^4 - 48*x^3 - 100*x^2 + 56*x + 23
```

Create your own

```
julia> Qx, x = QQ["x"]
(Univariate polynomial ring in x over QQ, x)

julia> NF, qr8 = number_field(x^4 - 8, "qurt(8)");

julia> real_embeddings(NF)
2-element Vector{AbsSimpleNumFieldEmbedding}:
  Real embedding with -1.68 of NF
  Real embedding with 1.68 of NF

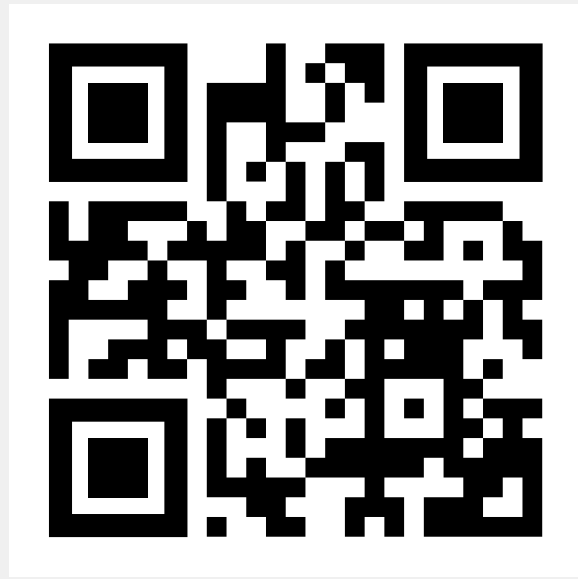
julia> ENF, qre8 = Hecke.embedded_field(NF, real_embeddings(NF)[2])
(Embedded field
  Number field of degree 4 over QQ
  at
  Real embedding with 1.68 of NF, qurt(8))

julia> P = convex_hull([1 0; -1 0; 0 qre8])
Polyhedron in ambient dimension 2 with EmbeddedAbsSimpleNumFieldElem
  type coefficients
```

Use the data outside of OSCAR

The data was also made available via Zenodo.

Via the JSON based MaRDI file format [3], the data can also be accessed by other means. A usable script for julia (without OSCAR) is provided.



Johnson data in Zenodo

References

- [1] Thomas Breuer, Michael Joswig, and Gunter Malle. Zeros of S -characters. *J. Comput. Algebra*, 13/14:Paper No. 100031, 6, 2025.
- [2] Wolfram Decker, Christian Eder, Claus Fieker, Max Horn, and Michael Joswig, editors. *The computer algebra system OSCAR—algorithms and examples*, volume 32 of *Algorithms and Computation in Mathematics*. Springer, Cham, [2025] ©2025.
- [3] Antony Della Vecchia, Michael Joswig, and Benjamin Lorenz. A FAIR file format for mathematical software. In *Mathematical software—ICMS 2024*, volume 14749 of *Lecture Notes in Comput. Sci.*, pages 234–244. Springer, Cham, [2024] ©2024.
- [4] Zoe Geiselmann, Alexej Jordan, Michael Joswig, Bernd Sturmfels, Marta Panizzut, and Olivia Röhrig. Exact johnson solids, February 2024.
- [5] Norman W. Johnson. Convex polyhedra with regular faces. *Canadian J. Math.*, 18:169–200, 1966.
- [6] A. V. Timofeenko. The non-Platonic and non-Archimedean noncomposite polyhedra. *J. Math. Sci.*, 162:710–729, 2009.
- [7] Viktor. A. Zalgaller. *Convex Polyhedra with regular Faces*. Consultants Bureau, New York, 1969.