

Mixed Volumes of Normal Complexes

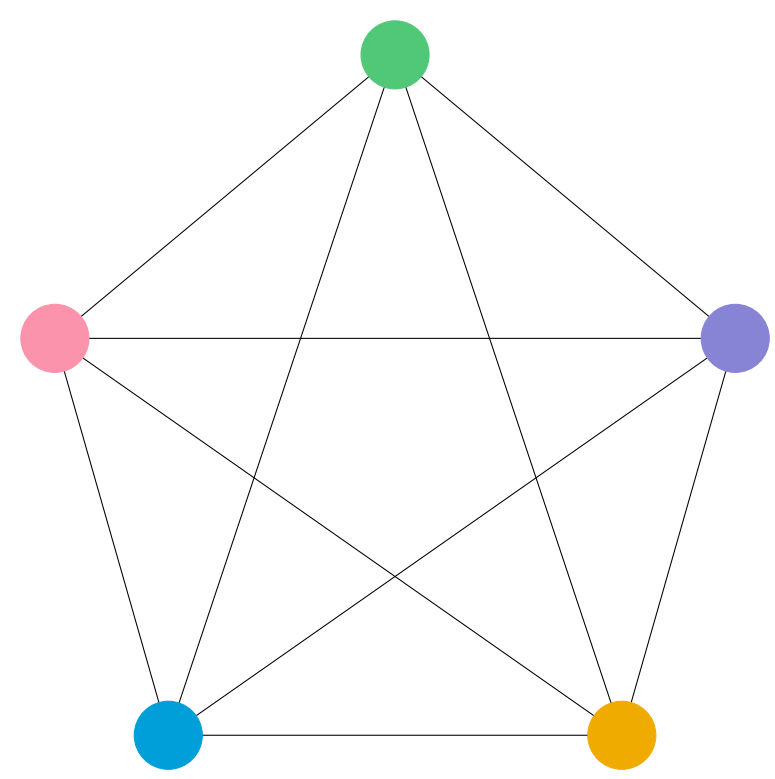
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Matroids and Characteristic Polynomials

Matroids are generalizations of the notion of independence. We are typically familiar with linear independence of sets of vectors or independence of edges of graphs; matroids generalize both. There exists a polynomial associated to each matroid, known as the **characteristic polynomial**, that generalizes the chromatic polynomial of a graph.



$$\chi_G(\lambda) = 1n^5 - 10n^4 + 35n^2 - 50n^2 + 24n$$

Conjecture [Heron-Rota-Welsh]

The coefficients of the characteristic polynomial of a matroid, $w_0, w_1, \dots, w_k$, form a log-concave sequence. More precisely

$$w_i^2 \geq w_{i-1}w_{i+1}$$

for $0 < i < k$.

Chow Rings

Also associated to a matroid is a graded ring known as a **Chow ring**. There exists a degree map from elements of maximal degree to the underlying field. The work of June Huh and Eric Katz showed that the coefficients of the characteristic polynomial appear as mixed degrees of certain divisors (linear elements of the Chow ring).

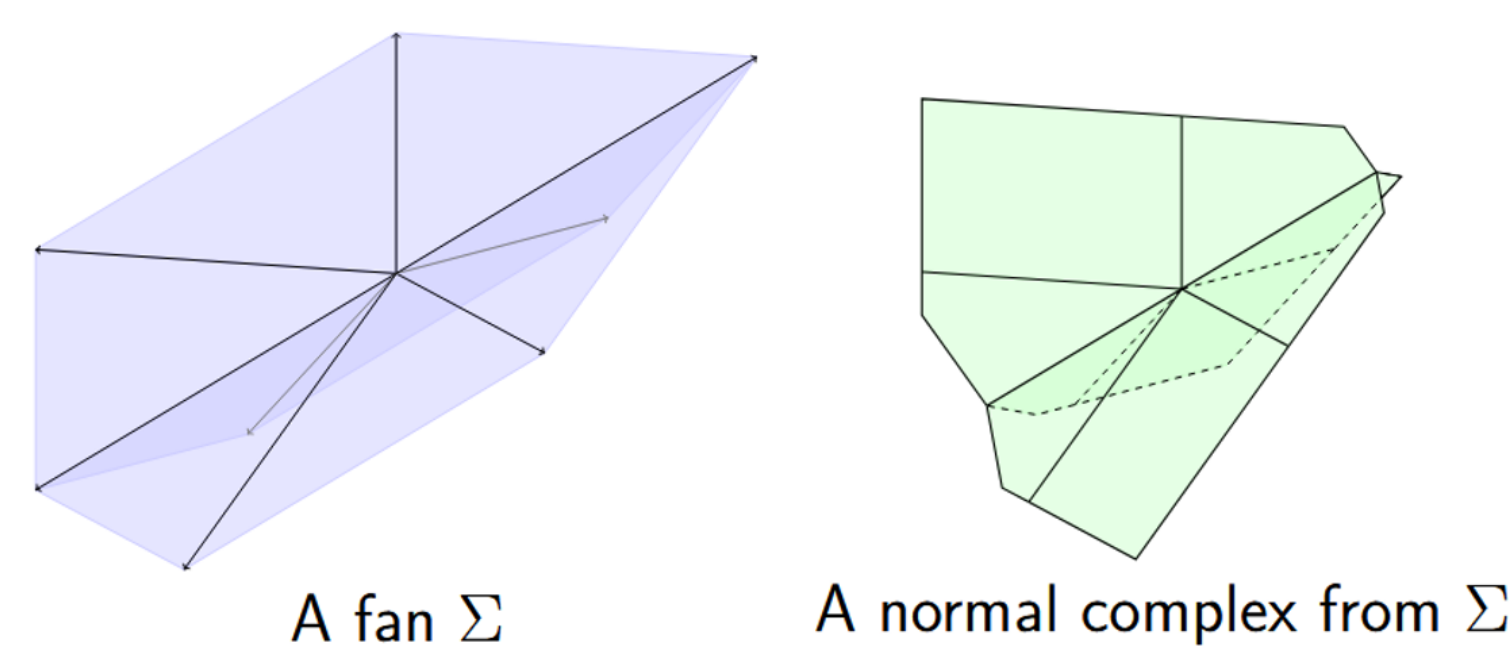
In 2018, Adiprasito, Huh, and Katz, showed that the Chow ring associated to a matroid satisfies a suite of properties allowing them to develop **combinatorial Hodge theory**. These properties allowed them to address the conjecture of Heron, Rota, and Welsh.

Theorem [Adiprasito-Huh-Katz]

The Heron-Rota-Welsh conjecture is true.

Normal Complexes

Given a simplicial fan, Σ , we can truncate the fan with hyperplanes normal to the 1-dimensional rays.



The resulting truncation is called a **normal complex** of Σ . The scalars defining the hyperplanes are collected into what we call the **z -vector**, with $z \in \mathbb{R}^{|\Sigma(1)|}$.

Theorem [Nathanson-Ross]

If Σ is a simplicial tropical fan, then the volume of a normal complex of Σ generated by a z -vector is equivalent to the divisor whose coefficients are given by the z -vector.

Mixed Volumes of Normal Complexes

In order to leverage the volumetric correspondence of normal complexes to *mixed* degrees of divisors, we develop a notion of mixed volumes of normal complexes, MVol_Σ . Given a fan Σ and z -vectors $z_1, \dots, z_r$ encoding r different truncations of the fan, we define the mixed volume of the r corresponding normal complexes

$$\text{MVol}_\Sigma(z_1, \dots, z_r)$$

as the sum, across all top-dimensional cones in Σ , of the mixed volume of the r polytopes associated to a given cone. We then show that $\text{MVol}_\Sigma(z_1, \dots, z_r)$ corresponds to mixed degrees of divisors in the Chow ring.

Further, we developed criteria to show when the mixed volumes of normal complexes obey the Alexandrov-Fenchel inequality, a property we call **AF**.

Theorem [Nowak-O’Melveny-Ross]

Let Σ be a simplicial tropical fan and $z_1, \dots, z_n$ be n different z -vectors. If

- ① every 2-dimensional face of Σ without the origin is connected and
- ② every 2-dimensional face of Σ is AF,

then Σ is AF. In particular, the Bergman fan of any matroid is AF.

Volume and the Degree Map

If Σ is a *complete* fan, then its corresponding toric variety $\mathcal{X}_\Sigma$ is complete. A result of toric geometry [Kushnirenko] tells us that there is a correspondence between the top power of the divisor whose coefficients are given by the z -vector and the volume of the normal polytope associated to Σ .

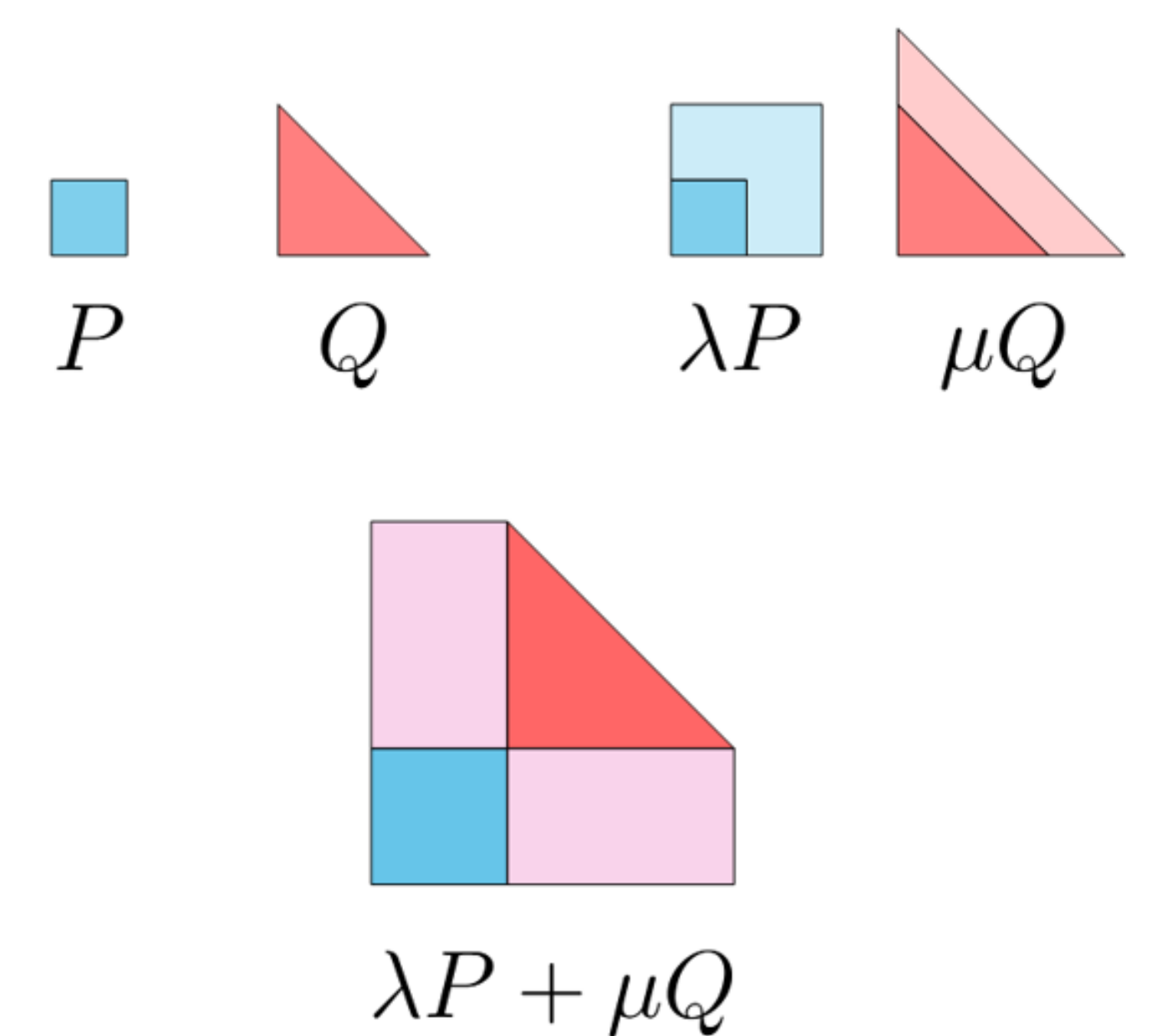
To any simplicial, but not necessarily complete, tropical fan, the work of Anastasia Nathanson and Dustin Ross developed a similar correspondence between the normal complex of the fan and the top degrees of divisors of the Chow ring associated to the fan.

Mixed Volumes

The **mixed volume function** is a map from an ordered multiset of convex bodies in $\mathbb{R}^n$ to $\mathbb{R}_{\geq 0}$, uniquely characterized by the following properties:

- $\text{MVol}(K, K, \dots, K) = \text{Vol}(K)$, for any convex body K ,
- MVol is symmetric in all arguments, and
- MVol is multilinear.

Another view is that, as shown by Minkowski, the volume of convex bodies behave as homogeneous polynomials under Minkowski addition. Under this interpretation, the mixed volumes appear as coefficients of a symmetrized volume polynomial.



$$\text{Vol}(\lambda P + \mu Q) =$$

$$\text{Vol}(P)\lambda^2 + \text{MVol}(P, Q)\lambda\mu + \text{Vol}(Q)\mu^2$$

Theorem [Alexandrov–Fenchel]

For convex bodies $P, Q, C_3, \dots, C_n$ in $\mathbb{R}^n$,

$$\text{MVol}(P, Q, C_3, \dots, C_n)^2 \geq \text{MVol}(P, P, C_3, \dots, C_n) \times \text{MVol}(Q, Q, C_3, \dots, C_n).$$

This implies that the sequence

$$\{\text{MVol}(\underbrace{P, \dots, P}_k, \underbrace{Q, \dots, Q}_{n-k})\}_{k=0}^n$$

is log-concave

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