

The background image is a blurred photograph of a lecture hall. In the foreground, several audience members' hands are raised, some holding pens. In the background, a presenter is visible at the front of the room, and a large screen displays a presentation. The overall atmosphere is one of an interactive academic session.

Posters and presentations

A poster presentation

- **Your project** in a streamlined visual form A0 or A1
- Very different from the written article!
- Conveys the **main ideas** to your audience
- Intended as a **conversation starter**
- Details and proofs should be omitted
- See newly posted poster resources on the course website

Exact Johnson solids in OSCAR

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Solids with regular faces

The following polytypes make up all 3-dimensional polytypes with regular faces:

- 5 Platonic solids: vertex- and facet-transitive,
- 13 Archimedean solids: vertex-transitive,
- Regular prisms and antiprisms,
- 92 Johnson solids: not vertex-transitive.

A polytype is vertex-transitive if all its vertices are equivalent under its symmetries.


The regular prism over a heptagon The regular antiprism over a heptagon

Johnson solids

A list of the 92 Johnson solids was computed by Johnson in 1966 [5] and proven to be complete by Zalgaller in 1969 [7].

Johnson solids can be created by e.g.

- Elongations and gyroelongations: adding a prism or antiprism,
- Augmentations: adding a pyramid,
- Diminishings: taking away a pyramid or cupola,

from bipyramids and biplanoloids or the other regular solids.

There are 9 elementary or non-composite Johnson solids.

The exact coordinates of most Johnson solids include non-rational algebraic numbers. Some of them use roots of complicated polynomials and are notoriously hard to compute.

We implemented the 92 non-rational ones in OSCAR [2], verifying or correcting existing data.

For the non-composite Johnson solid we followed Timofeev's computations [6].



The pentagonal cupola J50
Representable by a simple quadratic algebraic field extension over $\mathbb{Q}$.

$$\begin{aligned} & \left(\pm \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} \right) \\ & \left(\pm \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} \right) \\ & \left(\pm \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} \right) \\ & \left(\pm \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} \right) \end{aligned}$$



The gyroelongated square pyramidal J57b
Representable by a simple non-quadratic algebraic field extension over $\mathbb{Q}$.

$$\begin{aligned} & \left(\pm \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} \right) \\ & \left(\pm \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} \right) \\ & \left(\pm \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} \right) \\ & \left(\pm \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} \right) \end{aligned}$$



The sphenocorona J60b
Representable by a non-simple algebraic field extension over $\mathbb{Q}$.

$$\begin{aligned} & \left(\pm \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} \right) \\ & \left(\pm \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} \right) \\ & \left(\pm \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} \right) \\ & \left(\pm \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} + \frac{1}{2} \sqrt{\frac{5}{3}} \right) \end{aligned}$$

Hint: $w \approx 0.85$ is a root of the polynomial $f \in \mathbb{Q}[x]$

$$f(x) = 9x^4 - 48x^3 - 100x^2 + 56x + 23$$

and $h \approx 0.52$ is a root of $g \in \mathbb{Q}[y]$

$$g(y) = y^4 - y^2 - 1.$$

We used OSCAR's functionality of algebraic field extensions. In OSCAR non-simple and non-quadratic algebraic field extensions are possible.

Use the Johnson solids in OSCAR

```
julia> J86 = Johnson_solid(J86)
Polytype in ambient dimension 3 with
EmbeddedNumberField{Hecke.NR{AlgebraicNumberFieldElem}}
type coefficients

julia> vertices(J86) [7]
3-element PointVector{EmbeddedNumberField{Hecke.NR{AlgebraicNumberFieldElem}}{AlgebraicNumberFieldElem}}:
0 (0.00)
2/27*3 + 32/135*3 + 334/405*3 - 107/810 (0.79)
16/135*3 + 16/135*3 - 88/225*3 - 8/225*3 (-0.43)

julia> volume(J86)
(-13/6*3 + 17/30*3 + 653/180*3 + 133/180*3) (1.52)

julia> MILP = mixed_integer_linear_program(J86, [-1,1,1]; integer_variables = [3]);
(a = 1/2);

julia> solve_milp(MILP)
EmbeddedField
Relative number field of degree 2 over number field
at
Complex embedding corresponding to root 0.52 of relative number field

julia> defining_polynomial(number_field(J86))
y^2 + a^2 - 1

julia> defining_polynomial(base_field(number_field(J86)))
60x^4 - 48x^3 - 100x^2 + 56x + 23
```

Create your own

```
julia> Qx, x = QQ["x"]
Univariate polynomial ring in x over QQ, x)

julia> NF, q8 = number_field(x^4 - 8, "q8(8)");

julia> real_embeddings(NF)
2-element Vector{AlgebraicNumberField{AlgebraicNumberFieldElem}}:
Real embedding with -1.68 of NF
Real embedding with 1.68 of NF

julia> EMF, q8b = Hecke.embedded_field(NF, real_embeddings(NF)[2]);
EmbeddedField
Number field of degree 4 over QQ
Real embedding with 1.68 of NF, q8b(8)

julia> P = convex_hull([1 0; -1 0; 0 q8b])
Polyhedron in ambient dimension 2 with EmbeddedNumberField{AlgebraicNumberFieldElem}
type coefficients
```

Use the data outside of OSCAR

The data was also made available via Zenodo.

Via the JSON based MathJax file format [3], the data can also be accessed by other means. A usable script for Julia (without OSCAR) is provided.



Johnson data in Zenodo

References

- [1] Thomas Brice, Michael Joswig, and Corinna Vitz. *Zeroes of B-splines*. *J. Comput. Appl. Math.* 314:1-10, 2017.
- [2] Michael Joswig, Corinna Vitz, and Michael Joswig, editors. *The computer algebra system OSCAR: algorithms and examples, volume 32 of Algorithms and Computation in Mathematics*. Springer, Cham, 2022.
- [3] Anthony Datta, Michael Joswig, and Corinna Vitz. *A MathJax format for mathematical software*. In *MathJax for mathematical software*, pages 239-244. Springer, Cham, 2024.
- [4] Zoltan Kovács, Michael Joswig, Bernd Sturmfels, Marta Panizut, and Olivia Rohrig. *Exact Johnson solids*. February 2024.
- [5] Norman W. Johnson. *Convex polyhedra with regular faces*. *Canadian Math. Bull.* 12:105-123, 1969.
- [6] A.V. Timofeev. *The non-regular and non-Archimedean nonconvex polyhedra*. *J. Math. Sci.* 132:711-729, 2005.
- [7] Walter S. Zalgaller. *Convex polyhedra with regular faces*. *Cambridge Monographs on Mathematics*, New York, 1969.

- **Introduction or Motivation**—set up the question or problem and provide relevant history.
- **Background**—present needed terms and theorems.
- **Results and Justification**—state your results and any supporting proofs, derivations, or data.
- **Discussion or Further Research**—discuss implications or pose future questions.

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Some Do's

- Tell one clear story
- Visual thinking
- Strategic use of space
- The hierarchy principle
- Design for conversation

From *The math poster survival guide* by Brett Wick

Matroids and Characteristic Polynomials

Matroids are generalizations of the notion of independence. We are typically familiar with linear independence of sets of vectors or independence of edges of graphs; matroids generalize both. There exists a polynomial associated to each matroid, known as the **characteristic polynomial**, that generalizes the chromatic polynomial of a graph.


$$\chi_G(\lambda) = 1\lambda^5 - 10\lambda^4 + 35\lambda^3 - 50\lambda^2 + 24\lambda$$

Conjecture [Heron-Rota-Welsh]

The coefficients of the characteristic polynomial of a matroid, $w_0, w_1, \dots, w_k$, form a log-concave sequence. More precisely

$$w_i^2 \geq w_{i-1}w_{i+1}$$

for $0 < i < k$.

Chow Rings

Also associated to a matroid is a graded ring known as a **Chow ring**. There exists a degree map from elements of maximal degree to the underlying field. The work of June Huh and Eric Katz showed that the coefficients of the characteristic polynomial appear as mixed degrees of certain divisors (linear elements of the Chow ring).

In 2018, Adiprasito, Huh, and Katz, showed that the Chow ring associated to a matroid satisfies a suite of properties allowing them to develop **combinatorial Hodge theory**. These properties allowed them to address the conjecture of Heron, Rota, and Welsh.

Normal Complexes

Given a simplicial fan, Σ , we can truncate the fan with hyperplanes normal to the 1-dimensional rays.



A fan Σ A normal complex from Σ

The resulting truncation is called a **normal complex** of Σ . The scalars defining the hyperplanes are collected into what we call the **z-vector**, with $z \in \mathbb{R}^{[2(1)]}$.

Theorem [Nathanson-Ross]

If Σ is a simplicial tropical fan, then the volume of a normal complex of Σ generated by a z-vector is equivalent to the divisor whose coefficients are given by the z-vector.

Volume and the Degree Map

If Σ is a **complete** fan, then its corresponding toric variety X_Σ is complete. A result of toric geometry [Kushnirenko] tells us that there is a correspondence between the top power of the divisor whose coefficients are given by the z-vector and the volume of the normal polytope associated to Σ .

To any simplicial, but not necessarily complete, tropical fan, the work of Anastasia Nathanson and Dustin Ross developed a similar correspondence between the normal complex of the fan and the top degrees of divisors of the Chow ring associated to the fan.

Theorem [Nathanson-Ross]

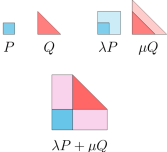
If Σ is a simplicial tropical fan, then the volume of a normal complex of Σ generated by a z-vector is equivalent to the divisor whose coefficients are given by the z-vector.

Mixed Volumes

The **mixed volume function** is a map from an ordered multiset of convex bodies in $\mathbb{R}^n$ to $\mathbb{R}_{\geq 0}$, uniquely characterized by the following properties:

- $\text{MVol}(K, K, \dots, K) = \text{Vol}(K)$, for any convex body K ,
- MVol is symmetric in all arguments, and
- MVol is multilinear.

Another view is that, as shown by Minkowski, the volume of convex bodies behave as homogeneous polynomials under Minkowski addition. Under this interpretation, the mixed volumes appear as coefficients of a symmetrized volume polynomial.



$\text{Vol}(\lambda P + \mu Q) = \text{Vol}(P)\lambda^2 + \text{MVol}(P, Q)\lambda\mu + \text{Vol}(Q)\mu^2$

Theorem [Alexandrov-Fenchel]

For convex bodies $P, Q, C_3, \dots, C_n$ in $\mathbb{R}^n$,

$$\text{MVol}(P, Q, C_3, \dots, C_n)^2 \geq \text{MVol}(P, P, C_3, \dots, C_n) \times \text{MVol}(Q, Q, C_3, \dots, C_n).$$

This implies that the sequence $\{\text{MVol}(\frac{P_{i-1} + \dots + P_i}{k}, \frac{Q_{i-1} + \dots + Q_i}{n-k})\}_{i=0}^n$ is log-concave

References

[1] K. Adiprasito, Huh, Bolyer theory for combinatorial geometries. 2020.

[2] Alexandrov. Zur theorie der gemischten volumina von konvexen korpern i. 1901.

[3] K. Huh. Log-concavity of characteristic polynomials and the hodge index of matroids. 2018.

[4] Kushnirenko. Polynome de nombre de zeroles de systeme lin. 1973.

[5] W. Nathanson. Tropical fans and normal complexes. 2019.

[6] L. Nowak, O'Melveny. Mixed volumes of normal complexes. 2022.

Some Dont's

- The wall of text trap
- Microscopic font syndrome
- Formula avalanche
- Assuming expert knowledge
- Color Chaos

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Matroids and Characteristic Polynomials

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$$\chi_G(\lambda) = 1n^5 - 10n^4 + 35n^3 - 50n^2 + 24n$$

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Chow Rings

Also associated to a matroid is a graded ring known as a **Chow ring**. There exists a degree map from elements of maximal degree to the underlying field. The work of June Huh and Eric Katz showed that the coefficients of the characteristic polynomial appear as mixed degrees of certain divisors (linear elements of the Chow ring).

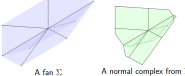
In 2018, Adiprasito, Huh, and Katz, showed that the Chow ring associated to a matroid satisfies a suite of properties allowing them to develop **combinatorial Hodge theory**. These properties allowed them to address the conjecture of Heron, Rota, and Welsh.

Theorem [Adiprasito-Huh-Katz]

The Heron-Rota-Welsh conjecture is true.

Normal Complexes

Given a simplicial fan, Σ , we can truncate the fan with hyperplanes normal to the 1-dimensional rays.



The resulting truncation is called a **normal complex** of Σ . The scalars defining the hyperplanes are collected into what we call the **z-vector**, with $z \in \mathbb{R}^{[2,1]}$.

Theorem [Nathanson-Ross]

If Σ is a simplicial tropical fan, then the volume of a normal complex of Σ generated by a z-vector is equivalent to the divisor whose coefficients are given by the z-vector.

Mixed Volumes of Normal Complexes

In order to leverage the volumetric correspondence of normal complexes to *mixed* degrees of divisors, we develop a notion of mixed volumes of normal complexes, MVol_Σ . Given a fan Σ and z-vectors $z_1, \dots, z_r$ encoding r different truncations of the fan, we define the mixed volume of the r corresponding normal complexes

$$\text{MVol}_\Sigma(z_1, \dots, z_r)$$

as the sum, across all top-dimensional cones in Σ , of the mixed volume of the r polytopes associated to a given cone. We then show that $\text{MVol}_\Sigma(z_1, \dots, z_r)$ corresponds to mixed degrees of divisors in the Chow ring.

Further, we developed criteria to show when the mixed volumes of normal complexes obey the Alexandrov-Fenchel inequality, a property we call **AF**.

Theorem [Nowak-O'Melveny-Ross]

Let Σ be a simplicial tropical fan and $z_1, \dots, z_n$ be n different z-vectors. If

- every 2-dimensional face of Σ without the origin is connected and
- every 2-dimensional face of Σ is AF,

then Σ is AF. In particular, the Bergman fan of any matroid is AF.

Volume and the Degree Map

If Σ is a **complete** fan, then its corresponding toric variety X_Σ is complete. A result of toric geometry [Kushnirenko] tells us that there is a correspondence between the top power of the divisor whose coefficients are given by the z-vector and the volume of the normal polytope associated to Σ .

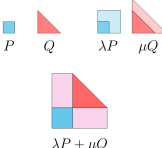
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- $\text{MVol}(K, K, \dots, K) = \text{Vol}(K)$, for any convex body K ,
- MVol is symmetric in all arguments, and
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Another view is that, as shown by Minkowski, the volume of convex bodies behave as homogeneous polynomials under Minkowski addition. Under this interpretation, the mixed volumes appear as coefficients of a symmetrized volume polynomial.


$$\text{Vol}(\lambda P + \mu Q) = \text{Vol}(P)\lambda^2 + \text{MVol}(P, Q)\lambda\mu + \text{Vol}(Q)\mu^2$$

Theorem [Alexandrov-Fenchel]

For convex bodies $P, Q, C_3, \dots, C_n$ in $\mathbb{R}^n$,

$$\text{MVol}(P, Q, C_3, \dots, C_n)^2 \geq \text{MVol}(P, P, C_3, \dots, C_n) \times \text{MVol}(Q, Q, C_3, \dots, C_n).$$

This implies that the sequence

$$\{\text{MVol}(\underbrace{P, \dots, P}_k, \underbrace{Q, \dots, Q}_{n-k})\}_{k=0}^n$$

is log-concave

References

- [1] K. Adiprasito, Buh, Bridgeland theory for combinatorial geometry.
- [2] Alexandrov, Sur l'inegalite des grandeurs volumiques de convexes ligenes n.
- [3] K. Buh, Log-concavity of characteristic polynomials and the bergman fan of matroids.
- [4] Kushnirenko, Polyèdres de newton et modules de newton.
- [5] D. Nathanson, Tropical fans and normal complexes.
- [6] D. Nowak, O'Melveny, Mixed volumes of normal complexes.

Test your poster

- **The 5 second test**

Can the topic/result be identified in 5 seconds?

- **The distance test**

Can parts be read/recognized from across the room?

- **The outsider test**

Can a general mathematician grasp what it is about and why it matters?

From *The math poster survival guide* by Brett Wick

Poster session

- Usually the most interactive event at a conference
- Equipped with coffee, tea, snacks!
- You will engage with people passing by
- Introduce yourself and your project
- Prepare an **adaptable introduction/pitch**
- Ask **and** receive questions

Presentation preparation aid

A – Introduction

B – Main message, outline and motivation

123 – Highlights with examples and discussion

C – Conclusion and future work

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The Science Library

University of Oslo

Poster resources

The math poster survival guide - Brett Wick

<https://www.math.wustl.edu/wp/wick/index.php/2025/10/19/the-math-poster-survival-guide-what-works-and-what-doesnt/>

University of Michigan Math Graduate - Posters

<https://sites.lsa.umich.edu/math-graduates/posters/>

Design and construction of mathematical posters - Anya Michaelson

<https://www.ams.org/notices/202002/rnoti-p189.pdf>

A blurred background image of a lecture hall. In the foreground, several students' hands are raised, some holding pens. In the background, a lecturer is visible, and a whiteboard with colorful diagrams is partially seen. The scene is brightly lit, likely from large windows on the right side of the frame.

Examples

The rank-nullity ring of a matroid

Tara Fife¹ Eline Mannino² Felipe Rincón¹

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Abstract

We introduce the *rank-nullity ring* of a matroid M . This is the subring of the Chow ring of the free matroid generated by sums of variables corresponding to subsets having the same rank and the same nullity in the matroid M . We present symmetric combinatorial expressions for the tautological Chern classes of M , introduced by Berget, Eur, Spink and Tseng which imply that they live in the rank-nullity ring of M .

Matroids and the Chow ring

A **matroid** on a ground set E of rank d is a function $r : 2^E \rightarrow \mathbb{Z}$ satisfying:

- 1 $0 \leq r(S) \leq |S|$
- 2 $S \subset U$ implies $r(S) \leq r(U)$
- 3 $r(S \cup U) + r(S \cap U) \leq r(S) + r(U)$

A **flat** is a subset $F \subseteq E$ such that $r(F \cup \{i\}) \geq r(F)$ for all $i \in E \setminus F$. The **Chow ring** [FY04] of a matroid is given by:

$$A^*(M) = \frac{\mathbb{Z}[x_F \mid \emptyset \subsetneq F \subseteq E]}{\langle x_F x_{F'} \mid F \not\subseteq F' \text{ and } F \not\supseteq F' \rangle + \langle \sum_{i \in F} x_F \mid i \in E \rangle}.$$

The rank-nullity ring of a matroid

The **uniform matroid** $U_{d,n}$ is given by:

$$r(S) = \begin{cases} |S| & \text{if } |S| \leq d \\ d & \text{else.} \end{cases}$$

Definition (Fife M. Rincón)

The **rank-nullity** ring R_M of a matroid M is given by:

$$R_M := \mathbb{Z}[y_{i,j} \mid i, j \in [0, n], i + j \leq n] \subseteq A^*(U_{n,n}),$$

where $y_{i,j} = \sum_{r(S)=i, \text{null}(S)=j} x_S$ and $\text{null}(S) = |S| - r(S)$.

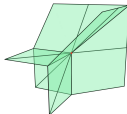
Chern numbers of matroids (motivation)

To a matroid we can associate the **Ardila-Klivans fan** $\text{AW}_*(M)$. The ring of **Minkowski weights** $MW_*(M)$ consists of weighted subfans satisfying the balancing condition.

Theorem ([AHK18]).

$$MW_*(M) \simeq A^*(M).$$

The **Chern numbers** [Man23] of a matroid are obtained by intersecting the **Chern-Schwartz-MacPherson cycles** [LRS20] of a matroid. These are elements of $MW_*(M)$. To better understand these numbers we want an expression for the CSM cycles of a matroid in $A^*(M)$.



$$\begin{array}{ccc} MW_k(M) & \xrightarrow{\simeq} & A^{d-k}(M) \\ \downarrow \psi & & \downarrow \psi \\ csm_k(M) & \longrightarrow & ? \end{array}$$

Bibliography

[FY04] Eva Marin Finkbeiner and Sergey Yuzvinsky. “Chow rings of toric varieties defined by atomic lattices”. In: *Inventiones mathematicae* 155.3 (2004), pp. 515–536.
[AHK18] Karim Adiprasito, June Hu, and Eyal Karu. “Bridgeland theory for combinatorial geometries”. In: *Journal of Mathematics* 139.2 (2018), pp. 381–402.
[LRS20] Lucinda Lopez de Medeiros, Felipe Rincón, and Irena Shipilova. “Chern-Schwartz-MacPherson cycles of matroids”. In: *Proceedings of the London Mathematical Society* 120.1 (2020), pp. 1–27.
[Ber+23] Andrew Berget et al. “Tautological classes of matroids”. In: *Inventiones mathematicae* (2023), pp. 1–95.
[Man23] Eline Mannino. “Chern numbers of matroids”. In: *arXiv preprint arXiv:2310.01950* (2023).

The Chern classes in the Chow ring

The CSM cycles are tightly related to a new geometric framework: the **tautological class of a matroid** [Ber+23] S_M , which unifies many matroid invariants. Its Chern classes satisfy

$$c_k(S_M) \in A^*(U_{n,n}).$$

Theorem ([Ber+23]).

$$csm_k(M) = c_{d-1-k}(S_M) \cdot \Delta_M,$$

where Δ_M is the class of the Ardila-Klivans fan.

Theorem (Fife M. Rincón)

The Chern classes $c_k(S_M)$ of a matroid are given by

$$c_k(S_M) = \sum_{\emptyset \subsetneq S_1 \subsetneq \dots \subsetneq S_k \subseteq E} \alpha_S(S_1^p, \dots, S_k^p) x_{S_1}^{p_1} x_{S_2}^{p_2} \dots x_{S_k}^{p_k},$$

where $\sum_{i=1}^k p_i = k$ and the sum is taken over all chains of subsets.

The coefficients are given by

$$\alpha_S(S_1^p, S_2^p, \dots, S_t^p) = (-1)^k \binom{r(S_1)}{p_1} \binom{r(S_2) - \tilde{p}_1}{p_2} \dots \binom{r(S_t) - \tilde{p}_{t-1}}{p_t}$$

where $\tilde{p}_i = \sum_{j=1}^i p_j$.

Corollary (Fife M. Rincón). The Chern classes $c_k(S_M)$ live in the rank-nullity ring R_M .

Properties of R_M

Proposition (Fife M. Rincón).

1. $R_M \subseteq A^*(U_{n,n})^{\text{Aut}(M)}$
2. $R_{U_{d,n}} = A^*(U_{n,n})^{S^n}$

Corollary (Fife M. Rincón)

The set

$\mathcal{B}_{R_{U_{d,n}}} = \{y_{i_1}^{p_1} y_{i_2}^{p_2} \dots y_{i_t}^{p_t} \mid i_j - i_{j-1} > p_j - p_{j-1} \text{ and } i_1 < i_2 < \dots < i_t\}$
is a basis for $R_{U_{d,n}}$

$$\left[\binom{n-1}{0}, \binom{n-1}{1}, \dots, \binom{n-1}{n-1} \right].$$

Proposition (Fife M. Rincón). The Hilbert function of R_M is non-decreasing up to $\lfloor \frac{n}{2} \rfloor$.



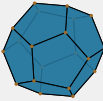


Solids with regular faces

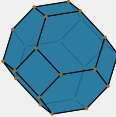
The following polytopes make up all 3-dimensional polytopes with regular faces:

- 5 Platonic solids: vertex- and facet-transitive,
- 13 Archimedean solids: vertex-transitive,
- Regular prisms and antiprisms,
- 92 Johnson solids: not vertex-transitive.

A polytope is *vertex-transitive* if all its vertices are equivalent under its symmetries.



The dodecahedron



The permutohedron



The regular prism over a hexagon



The regular antiprism over a hexagon

Johnson solids

A list of the 92 Johnson solids was computed by Johnson in 1966 [5] and proven to be complete by Zalgaller in 1969 [7].

Johnson solids can be created by e.g.

- Elongations and gyroelongations: adding a prism or antiprism,
- Augmentations: adding a pyramid,
- Diminishings: taking away a pyramid or cupola,

from (bilyramids and bilycupolas or the other regular solids).

There are 9 elementary or non-composite Johnson solids.

The exact coordinates of most Johnson solids include non-rational algebraic numbers. Some of them use roots of complicated polynomials and are notoriously hard to compute.

We implemented the 63 non-rational ones in OSCAR [2], verifying or correcting existing data.

For the non-composite Johnson solid we followed Timofeenko's computations [6].



The pentagonal cupola J(5):
Representable by a simple quadratic algebraic field extension over $\mathbb{Q}$.



The gyroelongated square pyramid J(10):
Representable by a simple non-quadratic algebraic field extension over $\mathbb{Q}$.



The sphenocorona J(86):
Representable by a non-simple algebraic field extension over $\mathbb{Q}$.

$$\begin{aligned} & \bullet \left(\pm \frac{1}{2} \sqrt{5} + \frac{1}{4} \sqrt{\frac{13}{60}} \sqrt{5} + \frac{1}{12} \right) \\ & \bullet \left(\pm \frac{1}{2} \sqrt{5} + \frac{1}{4} \sqrt{\frac{13}{60}} \sqrt{5} - \frac{1}{12} \right) \\ & \bullet \left(\pm \left(\frac{1}{4} \sqrt{5} + \frac{1}{4} \right) - \frac{1}{15} \sqrt{5} + \frac{1}{2} \sqrt{\frac{13}{60}} \sqrt{5} - \frac{5}{17} \right) \\ & \bullet \left(0, -\frac{1}{15} \sqrt{5}, \frac{1}{30} \sqrt{5} + \frac{1}{6} \right) \\ & \bullet \left(\pm \left(\frac{1}{2} \sqrt{5} + \frac{1}{2} \right) - \frac{1}{15} \sqrt{5}, \frac{1}{2} \sqrt{\frac{13}{60}} \sqrt{5} + \frac{1}{12} \right) \\ & \bullet \left(\pm \left(\frac{1}{4} \sqrt{5} + \frac{3}{4} \right) - \frac{1}{15} \sqrt{5} - \frac{1}{2} \sqrt{\frac{13}{60}} \sqrt{5} + \frac{1}{12} \right) \\ & \bullet \left(\pm \frac{1}{2}, -\frac{15}{60} \sqrt{5} - \frac{1}{4} \sqrt{\frac{13}{60}} \sqrt{5} + \frac{1}{12} \right) \end{aligned}$$
$$\begin{aligned} & \bullet \left(0, \pm \frac{1}{4} \sqrt{5} + \frac{1}{4} \sqrt{5} \right), \\ & \bullet \left(\pm \frac{1}{2}, \pm \frac{1}{2} \sqrt{\frac{13}{60}} \sqrt{5} \right), \\ & \bullet \left(0, \pm \frac{1}{4} \sqrt{5}, \pm \frac{1}{4} \sqrt{5} \right), \\ & \bullet \left(\pm \frac{1}{4} \sqrt{5}, \pm \frac{1}{4} \sqrt{5} \right) \end{aligned}$$
$$\begin{aligned} & \bullet \left(0, \pm \left(\frac{2}{27} a^3 + \frac{32}{135} a^2 + \frac{334}{405} - \frac{107}{405} \left(\frac{16}{15} a^3 + \frac{106}{75} a^2 - \frac{56}{225} a - \frac{83}{225} \right) b \right) \right) \\ & \bullet \left(\pm \frac{1}{2}, 0, \left(\frac{13}{27} a^3 + \frac{208}{135} a^2 - \frac{1733}{810} a - \frac{449}{805} b \right) \right) \end{aligned}$$

Here $a \approx 0.85$ is a root of the polynomial $f \in \mathbb{Q}[x]$

$$f(x) = 60x^4 - 48x^2 - 100x^2 + 56x + 23$$

and $b \approx 0.32$ is a root of $g \in \mathbb{Q}[y][y]$

$$g(y) = y^2 + a^2 - 1.$$

We used OSCAR's functionality of algebraic field extensions. In OSCAR non-simple and non-quadratic algebraic field extensions are possible.

Use the Johnson solids in OSCAR

```
julia> J86 = johnson_solid(86)
Polytope in ambient dimension 3 with
  EmbeddedNumFieldElem(Hecke.RelSimpleNumFieldElem(AbsSimpleNumFieldElem))
type coefficients

julia> vertices(J86) [7]
3-element PointVector{EmbeddedNumFieldElem(Hecke.RelSimpleNumFieldElem{
  AbsSimpleNumFieldElem})}:
 0 (0.00)
 2//27*a^3 + 32//135*a^2 + 334//405*a - 107//810 (0.79)
 (16//15*a^3 + 106//75*a^2 - 568//225*a - 83//225)*b (-0.43)

julia> volume(J86)
(-13//6*a^3 + 17//30*a^2 + 653//180*a + 133//180)*b (1.52)

julia> MILP = mixed_integer_linear_program(J86, [-1,1,1]; integer_variables = [3]);

julia> solve_milp(MILP)
((a + 1//2),
 EmbeddedNumFieldElem(Hecke.RelSimpleNumFieldElem(AbsSimpleNumFieldElem))
 [-a, 1//2, 0])

julia> F = coefficient_field(J86)
Embedded field
Relative number field of degree 2 over number field
at
Complex embedding corresponding to root 0.52 of relative number field

julia> defining_polynomial(number_field(F))
y^2 + a^2 - 1

julia> defining_polynomial(base_field(number_field(F)))
60*x^4 - 48*x^3 - 100*x^2 + 56*x + 23
```

Create your own

```
julia> QQ, x = QQ["x"]
(Univariate polynomial ring in x over QQ, x)

julia> NF, qr8 = number_field(x^4 - 8, "qrt(8)");

julia> real_embeddings(NF)
2-element Vector{AbsSimpleNumFieldElem}:
Real embedding with -1.68 of NF
Real embedding with 1.68 of NF

julia> ENF, qre8 = Hecke.embedded_field(NF, real_embeddings(NF)[2])
(Embedded field
Number field of degree 4 over QQ
at
Real embedding with 1.68 of NF, qrt(8))

julia> P = convex_hull([1 0; -1 0; 0 qre8])
Polyhedron in ambient dimension 2 with EmbeddedAbsSimpleNumFieldElem
type coefficients
```

Use the data outside of OSCAR

The data was also made available via Zenodo.

Via the JSON based MaRDI file format [3], the data can also be accessed by other means. A usable script for julia (without OSCAR) is provided.



Johnson data in Zenodo

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Mixed Volumes of Normal Complexes

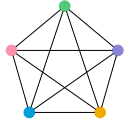
Lauren Nowak[†], Patrick O'Melveny[†], and Dustin Ross^{*}

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Matroids and Characteristic Polynomials

Matroids are generalizations of the notion of independence. We are typically familiar with linear independence of sets of vectors or independence of edges of graphs; matroids generalize both. There exists a polynomial associated to each matroid, known as the **characteristic polynomial**, that generalizes the chromatic polynomial of a graph.



$$\chi_G(\lambda) = 1n^5 - 10n^4 + 35n^3 - 50n^2 + 24n$$

Conjecture [Heron-Rota-Welsh]

The coefficients of the characteristic polynomial of a matroid, $w_0, w_1, \dots, w_k$, form a log-concave sequence. More precisely

$$w_i^2 \geq w_{i-1}w_{i+1}$$

for $0 < i < k$.

Chow Rings

Also associated to a matroid is a graded ring known as a **Chow ring**. There exists a degree map from elements of maximal degree to the underlying field. The work of June Huh and Eric Katz showed that the coefficients of the characteristic polynomial appear as mixed degrees of certain divisors (linear elements of the Chow ring).

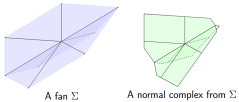
In 2018, Adiprasito, Huh, and Katz, showed that the Chow ring associated to a matroid satisfies a suite of properties allowing them to develop **combinatorial Hodge theory**. These properties allowed them to address the conjecture of Heron, Rota, and Welsh.

Theorem [Adiprasito-Huh-Katz]

The Heron-Rota-Welsh conjecture is true.

Normal Complexes

Given a simplicial fan, Σ , we can truncate the fan with hyperplanes normal to the 1-dimensional rays.



The resulting truncation is called a **normal complex** of Σ . The scalars defining the hyperplanes are collected into what we call the **z-vector**, with $z \in \mathbb{R}^{|\Sigma(1)|}$.

Theorem [Nathanson-Ross]

If Σ is a simplicial tropical fan, then the volume of a normal complex of Σ generated by a z -vector is equivalent to the divisor whose coefficients are given by the z -vector.

Mixed Volumes of Normal Complexes

In order to leverage the volumetric correspondence of normal complexes to *mixed* degrees of divisors, we develop a notion of mixed volumes of normal complexes, MVol_Σ . Given a fan Σ and z -vectors $z_1, \dots, z_r$ encoding r different truncations of the fan, we define the mixed volume of the r corresponding normal complexes

$$\text{MVol}_\Sigma(z_1, \dots, z_r)$$

as the sum, across all top-dimensional cones in Σ , of the mixed volume of the r polytopes associated to a given cone. We then show that $\text{MVol}_\Sigma(z_1, \dots, z_r)$ corresponds to mixed degrees of divisors in the Chow ring.

Further, we developed criteria to show when the mixed volumes of normal complexes obey the Alexandrov-Fenchel inequality, a property we call **AF**.

Theorem [Nowak-O'Melveny-Ross]

Let Σ be a simplicial tropical fan and $z_1, \dots, z_n$ be n different z -vectors. If

- every 2-dimensional face of Σ without the origin is connected and
- every 2-dimensional face of Σ is AF,

then Σ is AF. In particular, the Bergman fan of any matroid is AF.

Volume and the Degree Map

If Σ is a **complete** fan, then its corresponding toric variety $\mathcal{X}_\Sigma$ is complete. A result of toric geometry [Kushnirenko] tells us that there is a correspondence between the top power of the divisor whose coefficients are given by the z -vector and the volume of the normal polytope associated to Σ .

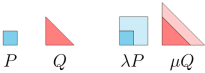
To any simplicial, but not necessarily complete, tropical fan, the work of Anastasia Nathanson and Dustin Ross developed a similar correspondence between the normal complex of the fan and the top degrees of divisors of the Chow ring associated to the fan.

Mixed Volumes

The **mixed volume function** is a map from an ordered multiset of convex bodies in $\mathbb{R}^n$ to $\mathbb{R}_{\geq 0}$, uniquely characterized by the following properties:

- $\text{MVol}(K, K, \dots, K) = \text{Vol}(K)$, for any convex body K ,
- MVol is symmetric in all arguments, and
- MVol is multilinear.

Another view is that, as shown by Minkowski, the volume of convex bodies behave as homogeneous polynomials under Minkowski addition. Under this interpretation, the mixed volumes appear as coefficients of a symmetrized volume polynomial.



$$\lambda P + \mu Q$$

$$\text{Vol}(\lambda P + \mu Q) =$$

$$\text{Vol}(P)\lambda^2 + \text{MVol}(P, Q)\lambda\mu + \text{Vol}(Q)\mu^2$$

Theorem [Alexandrov-Fenchel]

For convex bodies $P, Q, C_3, \dots, C_n$ in $\mathbb{R}^n$,

$$\text{MVol}(P, Q, C_3, \dots, C_n)^2 \geq \text{MVol}(P, P, C_3, \dots, C_n) \times \text{MVol}(Q, Q, C_3, \dots, C_n).$$

This implies that the sequence $\{\text{MVol}(\underbrace{P, \dots, P}_k, \underbrace{Q, \dots, Q}_{n-k})\}_{k=0}^n$ is log-concave

References

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Bernshtein–von Mises theorems for a class of Bayesian non-parametrics setups for stationary time series

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Introduction and summary

We construct a Bayesian nonparametric framework for estimation and inference for stationary Gaussian time series with expectation zero; de-trending can be used to deal with series with regression type of trends. The idea is to use the **frequency representation**

$$C(h) = 2 \int_0^\pi e^{-i\omega h} dF(\omega),$$

to build a **nonparametric envelope** around a parametric covariance function by putting a prior on a family of spectral measures. The general workflow can be summarised as

$$\begin{array}{ccc} \pi(C(\cdot)) & & \pi(C(\cdot) \mid \text{data}) \\ \downarrow & & \uparrow \\ \pi(F(\cdot)) & \longrightarrow & \pi(F(\cdot) \mid \text{data}) \end{array}$$

In particular, we establish **Bernshtein–von Mises type of results**, in the sense that the posterior has the same limit distribution as the empirical analogue in (1). To achieve this we will: (i) use a **piecewise constant prior**, and (ii) work under an alternative **contiguous measure**; both are needed to simplify the construction and associated derivations and conditions.

Nonparametrics inference

Let $Y_1, \dots, Y_n$ be a series generated by a model with spectral measure F° . The empirical

$$\hat{F}_n(t) = \frac{2\pi}{n} \sum_{j=0}^{[n/2]} I_n(2\pi j/n),$$

with $I_n(w) = |\sum_{t=1}^n Y_t \exp\{i\omega t\}|^2 / (2\pi n)$, has limit

$$\sqrt{n}(\hat{F}_n(t) - F^\circ(t)) \rightarrow_d W(2\pi \int_0^t f^\circ(\omega)^2 d\omega), \quad (1)$$

see e.g. Ibragimov (1963) for details.

The piecewise constant priors

Let $0 = w_0 < w_1 < \dots < w_{m_n} = \pi$ be a growing partition, then a **piecewise constant prior** on the spectral density (Figure 1) results in the prior

$$F_n(t) = \sum_{\{j: w_{j-1} \leq t\}} (w_{j+1} - w_j) f_j, \text{ with i.i.d. } f_j \sim \pi_j$$

on the spectral measures, for $t = w_j$ and with linear interpolation between these points.

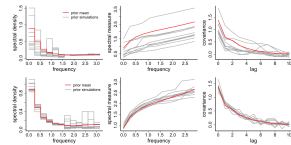


Figure 1: Samples from the prior with centred around an AR(2) with $\sigma = 1$ and $\rho = (0.3, 0.2)$ and two types of concentration.

Bernshtein–von Mises for time series

Let $B_n(t) = \sqrt{n}(F_n(t) - \hat{F}_n(t))$, then the goal is to obtain simple conditions that secure

$$B_n(t) \mid \text{data} \rightarrow_d W(2\pi \int_0^t f^\circ(\omega)^2 d\omega) \quad (2)$$

in probability.

Air pollution

As an illustration we consider hourly measurements of fine particulate matter ($\text{PM}_{2.5}$), a type of air pollutant referring to tiny particles, or droplets, in the air that are two and one half microns or less in width. Exposure to fine particles can cause short-term effects, like eye, nose, throat and lung irritation, long-term exposure can affect lung function and has also been shown to be related to asthma and heart disease. For health reasons, government regulations typically restrict the daily average emission to 25–35 $\mu\text{g}/\text{m}^3$.

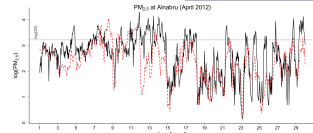


Figure 2: Observed and prognosis (made 24h in advanced) of $\text{PM}_{2.5}$ at Alnabru (Norway).

The Norwegian Institute for Air Research (NILU) in collaboration with the Norwegian Meteorological Institute (MET) makes prognosis/forecasts for different kinds of air pollution. In particular they make prognosis for $\text{PM}_{2.5}$ (Figure 2) using the current history and computer simulations. These forecast can be used to predict periods with potential high emissions, enabling the legislature to take appropriate actions in advanced. There are certain systematic biases in the prognosis, however, which may be corrected using a stochastic model.

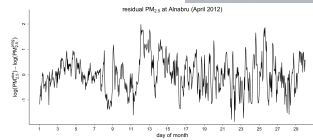


Figure 3: The difference between observed and prognosis (made 24h in advanced) of $\text{PM}_{2.5}$ (from Figure 2) seems to be sufficiently stationary.

Here, we will focus on the probability that at least one of three following $\text{PM}_{2.5}$ measurements will exceed the critical value of 25 $\mu\text{g}/\text{m}^3$, by constructing a posterior for

$$1 - \Pr\{Y_{n+1} \leq a, Y_{n+2} \leq a, Y_{n+3} \leq a \mid \text{data}_k\}$$

with $a = \log(25)$, $Y_i = \text{PM}_{2.5,i}^{\text{prog}} + \epsilon_i$ and $\text{data}_k = \{Y_n, \dots, Y_{n-k+1}\}$, by placing a nonparametric envelope around the simple AR(1) model (Figure 4).

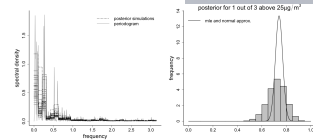


Figure 4: Empirical periodogram and samples from posterior (left), and posterior density for threshold probability and density based on mle for an AR(7) (AIC winner), normal approximation and delta method (right).

Contiguity

Let P_n and Q_n be two sequences of probability measures defined on the same measurable space $(\Omega_n, \mathcal{A}_n)$ for $n \geq 1$. Then P_n and Q_n are said to be mutually contiguous if, for every sequence of sets $A_n \in \mathcal{A}_n$, $P_n(A_n) \rightarrow 0 \Leftrightarrow Q_n(A_n) \rightarrow 0$.

Why Contiguity?

Working under an alternative measure may simplify derivations and conditions. As an example, let $\theta \in \mathbb{R}$ and $\pi(s \mid Y_1, \dots, Y_n)$ be the posterior of $s = \sqrt{n}(\theta - \hat{\theta}_{\text{ML}})$, then asymptotic normality of the posterior follows if

$$\int_{\mathbb{R}} |\pi(s \mid Y_1, \dots, Y_n) - \kappa \phi(s\kappa)| ds \rightarrow 0 \quad (3)$$

in P_n -probability, where $\phi(\cdot)$ is the standard normal. Then, if P_n and Q_n are contiguous, and Q_n is sufficiently simpler, we may benefit by working under Q_n to also prove (3) in P_n -probability.

From dependence to independence

If P_n is the probability measure associated with the series $Y_1, \dots, Y_n$, it follows from Choudhuri et al. (2004) that there exist a **contiguous measure** Q_n (Figure 5) which is such that

$$\pi(F_n \mid Y_1, \dots, Y_n) = \pi(F_n \mid I_n(\omega_1), \dots, I_n(\omega_{m_n}))$$

under Q_n , where $I_n(\omega_j) \sim f^\circ(\omega_j)E_j$ are independent and $E_j \sim \text{Exp}(1)$.

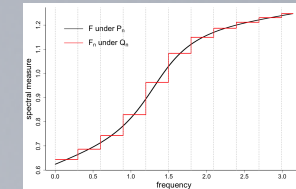


Figure 5: The spectral measures of the two contiguous measures P_n and Q_n . In particular, the log-likelihood under Q_n is exactly equal to the so-called Whittle approximation.

Conditions for Bernshtein–von Mises

Therefore, by working under Q_n (as illustrated in Figure 5) we may apply results from Hermansen and Hjort (2015) to derive simple conditions for Bernshtein–von Mises type of results for time series. In addition to standard regularity assumptions, we need that **the number of windows m_n** in the piecewise constant prior (Figure 1) satisfies

$$m_n / \sqrt{n} \rightarrow 0 \quad \text{and} \quad \sqrt{n} / m_n^2 \rightarrow 0$$

in order to secure the limit (2).

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Integrative clustering of high-dimensional data with joint and individual clusters

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Abstract

It is now possible to measure a range of genomic data types or data layers in a single tissue sample, e.g. gene expression, copy number variations, miRNA or mutation markers, and clustering of patient samples benefits from an integrative approach using all available data types. Earlier methods, however, are restricted to only allowing a joint cluster structures, equal in all data layers. We present a clustering extension of the Joint and Individual Variance Explained (JIVE) algorithm enabling construction of both joint and data type-specific clusters simultaneously (Hellton and Thoresen, 2016).

Method

Let $X_1, \dots, X_M$ be M genomic data types measured on the same $j = 1, \dots, n$ patients, concatenated into a single matrix

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_M \end{bmatrix},$$

of dimension $(p_1 + \dots + p_M) \times n$. Joint and Individual Clustering (JIC) assumes latent common and data type-specific cluster indicators, Z and $Z_1, \dots, Z_M$,

$$X_1 = W_1 Z + V_1 Z_1 + \varepsilon_1,$$

$$\vdots$$

$$X_M = W_M Z + V_M Z_M + \varepsilon_M,$$

with $\varepsilon_m \sim N(0, \sigma_m^2 I)$ for $m = 1, \dots, M$ and joint and individual loading matrices $W = [W_1, \dots, W_M]^T$ and $V_1, \dots, V_M$.

For given numbers of clusters K and $K_1, \dots, K_M$, the clusters are estimated by a two-stage procedure based on the singular value decomposition (SVD):

1. Estimate the joint and individual matrices, WZ and $V_m Z_m$ using the JIVE algorithm (Lock et al. 2013). Initialize $X^{\text{JOINT}} = X$ and repeat the following steps until convergence,
 - set WZ to be the SVD of X^{JOINT} with rank $r = K - 1$,
 - for $m = 1, \dots, M$, set $V_m Z_m$ to be the SVD of $X_m^{\text{INDIVID}} (I - ZZ^T)$ with rank $r_m = K_m - 1$, where $X_m^{\text{INDIVID}} = X_m - W_m Z$,
 - concatenate the matrices $X^{\text{JOINT}} = [X_1 - V_1 Z_1, \dots, X_M - V_M Z_M]^T$.

After convergence, Z is given by the r first right singular vectors of X^{JOINT} and Z_m is given by the r_m first right singular vectors of X_m^{INDIVID} for $m = 1, \dots, M$.

2. Cluster the rows of Z^T into K groups and the rows of Z_m^T into K_m groups for $m = 1, \dots, M$ using k-means clustering.

Selecting the numbers of clusters

The numbers of clusters, K and $K_1, \dots, K_M$, are chosen by assessing the number of non-normally distributed components in the concatenated and original data types, separately, under the assumption of normally distributed errors:

1. Assess the number of relevant components E in X : check the normality of the i th component score vector of X for increasing i , until the last non-normally distributed component is found and set E to the component number.
2. Assess the number of relevant subspaces E_m in X_m : For each $m = 1, \dots, M$, check the normality of the i th component score vector of X_m for increasing i up to E , until the last non-normally distributed component is found, and set E_m to the component number.
3. Based on $E, E_1, \dots, E_M$, calculate the ranks $r, r_1, \dots, r_m$ by

$$r = \frac{E_1 + \dots + E_M - E}{M - 1}, \quad r_m = E_m - r, \quad m = 1, \dots, M.$$

such that the numbers of clusters are $K = r + 1$ and $K_m = r_m + 1$.

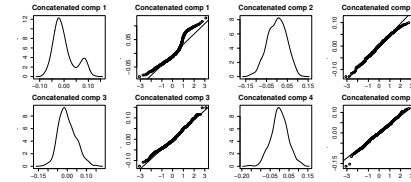
With the connection between the number of clusters and the rank of the latent structures (Ding and He, 2004), we check whether a new cluster is separated out in each added component until the first component with only a single cluster is found. When assuming normally distributed noise, it is possible to identify overlapping clusters by evaluating density and normal quantile-quantile plots. If the distribution of component scores deviates from normality, the component does not represent noise and clusters should be present.

Conclusion

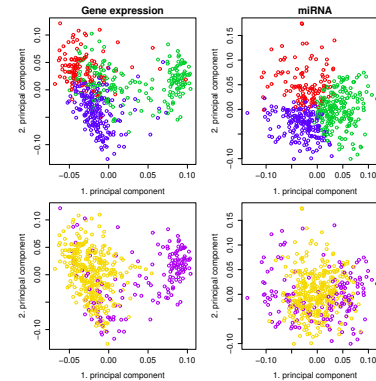
The Joint and Individual Clustering makes it possible to decompose data into common and data type-specific clusters following the JIVE algorithm. Example data from the Cancer Genome Atlas consisting of gene expression and miRNA from breast cancer tissues illustrates how there may exist independent groups only found in a single data type, in addition to cancer subtypes found jointly in different data types.

Example: gene expression and miRNA

The method is illustrated using data from The Cancer Genome Atlas (TCGA); gene expression ($p_1 = 1000$) and miRNA ($p_2 = 193$) measured in 500 breast cancer tissue samples. The density and qq-plots for the concatenated data (shown below) suggest that the three first components are not normally distributed, while the original expression and miRNA data suggest that the three and two first components deviate from normality, respectively.



This results in three joint clusters, $K = 3$, two expression-specific clusters, $K_1 = 2$, and no miRNA-specific clusters, $K_2 = 1$.



Coloring the observations according to the three joint clusters (upper panels) shows that the joint clusters are reflected in both the original data sources, while coloring the observations according to the two expression-specific clusters (lower panels) demonstrates how they only reflect the structure of expression data and not the structure of miRNA. Differences in survival between the different cluster configurations were not significant due to a low number of events.

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When did author B take over for author A? Confidence Distributions for Change-Points

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Introduction

The 15th century Catalan chivalry romance *Tirant lo Blanch* is an important work in medieval literature, and a perfect subject for a change-point analysis. The first author, Joanot Martorell, died before the completion of the novel and his friend and fellow knight, Marti Joan de Galba, took over the task. Many analyses, both statistical and literary, have attempted to identify exactly where the change-of-author takes place. Here, we propose our solution, using a new method which both estimates the change-point and provides the associated confidence curve.

Data

Our data is the text of an ancient Catalan book with 487 chapters and we search for the chapter where author B takes over from author A. From the text we can measure, for all chapters $i = 1, \dots, 487$:

Our data is the text of an ancient Catalan book with 487 chapters and we search for the chapter where author B takes over from author A. From the text we can measure, for all chapters $i = 1, \dots, 487$:

- y_i = average word length in chapter i , m_i = number of words in chapter i
- z_i = the vector of proportions of words of length 1 to 10 in chapter i .

For example $z_1 = (0.08, 0.23, 0.17, 0.08, 0.13, 0.08, 0.06, 0.07, 0.04, 0.07)$, so 8% of the words in chapter 1 are one-letters words (length 1), and 7% of the words have more than 10 letters.

General method

We denote the unknown change-point by τ . To estimate τ and the associated uncertainty we use a method described in Cunen, Hermansen and Hjort (2016). In this case, we assume independent data y_i , following a parametric model with

$$\begin{aligned} Y_i &\sim f(y, \theta_L) & \text{for } i \leq \tau \\ Y_i &\sim f(y, \theta_R) & \text{for } i \geq \tau + 1 \end{aligned}$$

We can then form the **profile log-likelihood function**

$$\begin{aligned} \ell_{\text{prof}}(\tau) &= \max_{i \leq \tau} \{\ell(\tau, \theta_L, \theta_R) : \text{all } \theta_L, \theta_R\} \\ &= \sum_{i \leq \tau} \log f(y_i, \hat{\theta}_L(\tau)) + \sum_{i \geq \tau+1} \log f(y_i, \hat{\theta}_R(\tau)). \end{aligned}$$

The maximiser $\hat{\tau}$ of this function is a good estimate of the change-point. From this we form the **deviance** function

$$D(\tau, Y) = 2\{\ell_{\text{prof}}(\hat{\tau}) - \ell_{\text{prof}}(\tau)\}.$$

In order to find the full **confidence curve** (essentially confidence intervals at different levels) we need the estimated distribution of $D(\tau, Y)$ for each position τ ,

$$\text{cc}(\tau) = \Pr_{\tau, \hat{\theta}_L, \hat{\theta}_R} \{D(\tau, Y) < D(\tau, y_{\text{obs}})\}.$$

This can be computed via stochastic simulation: we generate a large number B of simulated datasets Y^* from $f(y, \hat{\theta}_L)$ to the left of τ and $f(y, \hat{\theta}_R)$ to the right, at each candidate value τ , and calculate

$$\text{cc}(\tau) = \frac{1}{B} \sum_{j=1}^B I\{D(\tau, Y_j^*) < D(\tau, y_{\text{obs}})\}.$$



Conclusion and references

Using word lengths as a measure of literary style, we have discovered two clear candidates for the chapter where Marti Joan de Galba took over from Joanot Martorell. Both our analyses indicate that the change in *Tirant lo Blanch* takes place in the last quarter of the book. This is consistent with previous studies, both statistical and literary.

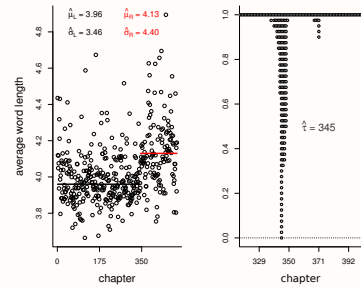
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Cunen, C., Hermansen, G. and Hjort, N.L. (2016). Confidence distributions for change-points and regime shifts. *To appear in Journal of Statistical Planning and Inference*.
Riba, A. & Ginebra, J. (2005). Change-point estimation in a multinomial sequence and homogeneity of literary style. *Journal of Applied Statistics* 32, 61–74.

Average word length

For the average word length in each chapter we assume the following model,

$$\begin{aligned} y_i &\sim N(\mu_L, \sigma_L^2/m_i) & \text{for } i \leq \tau \\ y_i &\sim N(\mu_R, \sigma_R^2/m_i) & \text{for } i \geq \tau + 1. \end{aligned}$$

According to this model, our best guess is that the change of author takes place in **chapter 345**. Author B uses longer words on average, and is also more variable across chapters. (The mean number of words per chapter is $\bar{m} = 927$.)

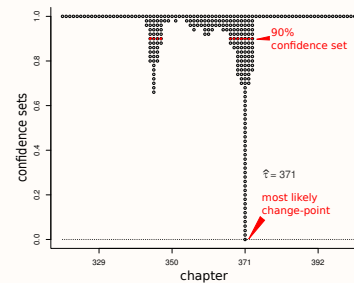


Proportion of words of different length

For the vector of proportions of words of length 1 to 9 (disregarding element no. 10, since the proportions sum to one) we assume the following multinomial distribution,

$$\begin{aligned} z_i &\sim N_9(\xi_L, \Sigma_L/m_i) & \text{for } i \leq \tau \\ z_i &\sim N_9(\xi_R, \Sigma_R/m_i) & \text{for } i \geq \tau + 1. \end{aligned}$$

The estimated mean vectors indicate that author B differs from author A in using more one-letter words and also using more long words (with more than 7 letters). The change-of-author point-estimate according to this model is **chapter 371**, but the model also places some confidence in the change taking place earlier, in chapter 345.



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The template

My Bachelor Project Title

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¹Department of Mathematics

Introduction

Here is where you can set up the question or problem you studied in your project and set that problem into context. This is also the place to provide any history or motivation. You may or may not want to use "Introduction" as the name here. Can you think of something more helpful or catchy?

Background

Here is where you could provide the main definitions and some of the base results you use in your project (not yet the highlights) think of this as a streamlined 'preliminaries' section which you might have in your final written project.

You can also use latex environments like "Theorem" and "Definition" to highlight your main points. These environments follow the color scheme of "block" as defined in the .cls file. Here is an example.

Definition

A finite projective plane consists of a finite set P and a collection of subsets of P denoted $\mathcal{L}$ satisfying three axioms.

Example I - The Fano plane



Figure 1: A depiction of a finite project plane with 7 points which are given by the dots. The set $\mathcal{L}$ consists of the 7 triplets of points which are connected by "lines".

Results and justification

This could be a section where you state your main results. If you decide to have any proofs on your poster they could also go here. Consider including a proof by example if there is an illustrative and not too technical one you could come up with.

Theorem

A finite projective plane over $\mathbb{F}_q$ contains $q^2 + q + 1$ points and $q^2 + q + 1$ lines.

Example II

The Fano plane from Example I is a finite projective plane over $\mathcal{F}_2$. It contains $7 = 2^2 + 2 + 1$ points and also 7 lines.

A finite projective plane has **order** q if it has $q^2 + q + 1$ points.

Discussion and future research

Not all finite projective planes arise from geometries over finite fields. However, all known examples have order q/p^n . The following is known as the **prime power conjecture**.

Conjecture

Finite projective planes only exist for $q = p^n$ where p is a prime.

References

Include a few of the main references you used using bibtex.

Acknowledgements

This project was prepared as part of MAT2000 - Project work in mathematics. I would like to thank my project supervisor **NAME**.



