

# Ramanujan matrixes from interlacing polynomials

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## Goal

- Low-level: understand the application of interlacing polynomials to spectral graph theory
- High-level: witness an interesting relation between combinatorics, probability theory, and linear algebra through this reading

## Description

This is a reading project on the application of interlacing polynomials to spectral graph theory, in particular, the Ramanujan graphs. *Expander graphs* form an important class of examples in the spectral graph theory, which studies various properties of graphs through the eigenvalues of their adjacency matrices. Let  $\Gamma$  be a  $d$ -regular graph, that is, each vertex is connected to  $d$  edges. If we take its adjacency matrix  $A_\Gamma$  and its eigenvalues  $\lambda_1(A_\Gamma), \lambda_2(A_\Gamma), \dots$  arranged in the decreasing order, we have the largest eigenvalue  $\lambda_1(A_\Gamma) = d$  (from knowing that  $(1, \dots, 1)^T$  is an eigenvector of  $A_\Gamma$ ), while the second eigenvalue  $\lambda_2(A_\Gamma)$  is related to the *Cheeger constant*  $h(\Gamma)$  of  $\Gamma$ , which gives an estimate on the size of subsets of vertices and their boundary sets.

One fundamental result of the theory (the Alon–Boppana bound) says that, if  $(\Gamma^{(i)})_{i=1}^\infty$  is a sequence of  $d$ -regular graphs, then we have  $\liminf_{i \rightarrow \infty} \lambda_2(A_{\Gamma^{(i)}}) \geq 2\sqrt{d-1}$ . We say that  $(\Gamma^{(i)})_i$  is a sequence of *Ramanujan graphs* if we have  $\lambda_2(A_{\Gamma^{(i)}}) \leq 2\sqrt{d-1}$ , so that the second largest eigenvalue  $\lambda_2(A_{\Gamma^{(i)}})$  stays at the optimal level. Such graphs is a class of expander graphs, and represent mathematical models of network systems that are resilient to cutting.

It turned out that constructing examples of Ramanujan graphs is a highly nontrivial task, and the first constructions due to Margulis and Lubotzky–Phillips–Sarnak relied on deep connections between eigenvalues of graphs coming from groups and geometry behind them, which forced  $d$  to be of the form  $p^k + 1$  for a prime number  $p$ . In a remarkable paper [MSS15], Marcus, Spielman, and Srivastava found a new way to construct Ramanujan graphs for arbitrary  $d$  based on the theory of interlacing polynomials, which look at real polynomials  $f(t)$  and  $g(t)$  such that their roots  $\lambda_1, \dots, \lambda_k$  and  $\mu_1, \dots, \mu_{k-1}$  form an interlacing sequence, that is,  $\lambda_1 > \mu_1 > \lambda_2 > \dots$ .

Our goal is to read a recent survey [Sri24] by Srivastava, one of the authors of [MSS15] to understand the main ideas behind the proof. While this survey offers a good outline of the theory, we will take up extra materials such as [MSS15] in order to pick up necessary details.

## References

- [MSS15] Adam W. Marcus, Daniel A. Spielman, and Nikhil Srivastava, *Interlacing families I: Bipartite Ramanujan graphs of all degrees*, Ann. of Math. (2) **182** (2015), no. 1, 307–325.
- [Sri24] Nikhil Srivastava, *Ramanujan graphs and interlacing families*, lecture at International Congress of Basic Science 2024. (arXiv:2412.20721)