

MAT2000 PROJECT: THE BANACH-TARSKI PARADOX AND AMENABILITY

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The Banach-Tarski paradox states that a solid ball in three-dimensional space can be decomposed into a finite number of disjoint subsets and be reassembled into two identical copies of the original ball. This result can be studied from the point of view of geometric group theory: It essentially boils down to the fact that $\mathbb{F}_2$, the free group on two generators, is a subgroup of the group of Euclidean transformations on three-dimensional space.

The above is a special example of paradoxical group actions, and we say that a group is paradoxical if its left action on itself is a paradoxical action.

Opposite to paradoxical groups are the amenable groups. These are of paramount importance in geometric group theory, as they form a significantly more tractable class of groups for many problems.

In the project we will first understand the Banach-Tarski paradox from a group-theoretic perspective, and take this as motivation for defining amenability. From here, there are many directions the project could go. We can for example investigate the class of *elementary amenable groups*, which are the amenable groups we can build from the “basic building blocks” of finite groups and the integers. It turns out that not all amenable groups are elementary amenable, and a natural problem would be to understand why this is.

Excellent sources for this material are

- “Amenability of discrete groups by examples” by Juschenko
- “Geometric Group Theory” by Drutu and Kapovich

both of which are available for free online.

Prerequisites: You should have taken a course in abstract algebra covering basic group theory, for example MAT2200 Groups, rings and fields